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Sandpile models:
The infinite volume model, Zhang's model and limiting shapes

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About the cover (design: Frank Fey):

The cover shows pictures of the double router model introduced in Chapter 5. The large picture on the front shows the direction function of the model with $h = 0$ *and* $n = 366,600$*, with all initial arrows horizontal (black). On the back, the central picture shows the according particle function, and the two small pictures are the direction functions for the model with* $h = -5$*,* $n = 400,000$ *(top), and* $h = -1$*,* $n = 100,000$ *(bottom). The according particle functions can be found in Figure 5.2 of this thesis.*

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

VRIJE UNIVERSITEIT

Sandpile models:
The infinite volume model, Zhang's model and limiting shapes

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
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in het openbaar te verdedigen
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door

Anna Cathérine Fey-den Boer

geboren te Amsterdam

promotor:	prof.dr. R.W.J. Meester
copromotor:	dr. F. Redig

aan Beppe,
tante Nel,
en vele andere sterke vrouwen

Contents

Chapter 1

Introduction

Sandpile models form a class of models with the possibility of endless variation, receiving interest from mathematical physicists [45, 47], experimentalists [1], probabilists [32], combinatorists [9] and algebraists [41]. The research of this thesis is probabilistic in nature; it ranges from the investigation of self-organized criticality (Chapters 2 and 3), to "quasi-units" in the stationary distribution of Zhang's model (Chapter 4), and limiting shapes of sandpile-related growth models (Chapter 5).

The content of Chapters 2 through 5 of this thesis consists of the text of four research articles, published or submitted for publication in various mathematical journals. Only in Chapter 2, I made some minor changes, because this was the first publication and meanwhile we changed some notations. The language in these chapters is very formal and specialistic. Here I attempt to present a more easily readable introduction of the models considered in this thesis.

1.1 Abelian sandpile basics

The abelian sandpile model is a mathematical model, rather than an accurate model of physical sand. Its origin is the first sandpile model proposed by Bak, Tang and Wiesenfeld [5] in 1988, who were inspired by the behavior of dry sand sprinkled on a flat surface. Think for example of sand in a hourglass. The sand tends to arrange itself in a pile that always has approximately the same slope. Indeed, if one sprinkles more sand on the top of such a pile, it does not become steeper, but instead an avalanche of sand occurs, after which the pile has again the same slope. The sand seems to organize itself into a "critical" state, that is, it tends to settle on the edge of stability. Incidentally, sand is not the best material to observe such a critical state experimentally; rice, that consists of larger and more elongated grains, forms a constant slope on a much larger scale and features much better observable avalanches [1].

Bak, Tang and Wiesenfeld numerically studied a discrete grid sandpile model in which the movement of sand grains is triggered by the local slope. Their model was suitable for numerical simulations. However, the model becomes much more mathematically tractable (but physically less realistic) if the movement of sand grains is defined to be triggered by the local height, irrespective of local slope, and thus capable of defying gravity. This is the abelian sandpile model. In retrospect, this model turns out to have been introduced by Engel in 1975 [18], as the probabilistic abacus.

To introduce the abelian sandpile model, we start with an example. Depicted below is a finite part of the two-dimensional grid $\mathbb{Z}^2$, with at every site a number. We call this a *configuration*. The number we interpret as number of sand grains, or height. Every site is stable, that is, the number is strictly less than 4 (more generally, in d dimensions, the number is less than $2d$). A configuration with only stable sites, such as this example, is a stable configuration.

2	3	1
3	3	0
1	2	3

The model evolves in discrete time. At the beginning of a time step, a grain of sand is added to a random site. Every site is equally likely to be chosen. In our example, the center site was chosen, and the configuration now looks like this:

2	3	1
3	4	0
1	2	3

The central site is not stable, and therefore it will *topple*. In a toppling, the site gives a grain of sand to each of its neighbors. The result is:

2	4	1
4	0	1
1	3	3

Now there are two unstable sites, so both will topple. Since they are at the boundary, they have only three neighbors, but they still lose four grains. So in a toppling of a boundary site, one grain is lost. The new result is:

4	0	2
0	2	1
2	3	3

Now a corner site is unstable. It will topple, and in its toppling two grains are lost. After this toppling, the configuration is stable again, so the final result is:

0	1	2
1	2	1
2	3	3

We call the total of all topplings in a time step an *avalanche*. The model is called *abelian* because the order of additions and/or topplings does not influence the final result, as is formally proved in [39]. What this means is the following: In the above example, we chose a certain order of topplings of unstable sites, namely, to topple every unstable site in parallel, and keep doing this until there are no unstable sites left. We can keep a record of how many times each site toppled, and to which site we added; we may even keep a cumulative record during several additions. Abelianness means that knowing the initial configuration and this record of additions and topplings, suffices to determine the final configuration, even though the record does not contain information on the order of additions and topplings.

In the example, the addition occurred to the center, and the four top left sites each toppled once. By abelianness, we are free to change this order, for instance, we could topple the corner site first. It would have a negative height for a while (since this feels slightly disturbing, we call such a toppling "illegal", and generally don't allow it), but after all other topplings, it would have height 0 again.

At this point I introduce some notation. We call η a configuration, that is, η is a vector that contains a height for every site. ξ is another configuration, obtained from η by performing topplings. T is another vector, containing for every site the number of these topplings. Δ is the toppling matrix, containing the information of how heights change in a toppling. We already know that if a certain site x topples, then site x loses $2d$ grains and every neighbor of x gains one. In matrix notation, we express this as $\Delta_{x,x} = 2d$, and $\Delta_{x,y} = -1$, for every y that is a neighbor of x. For all other y, $\Delta_{x,y} = 0$. The matrix $-\Delta$ is also known as the discrete Laplacian, or discrete second derivative, and furthermore, $-\frac{1}{2d}\Delta + I$ is the transition matrix for simple random walk.

We can now write

$$\eta - \Delta T = \xi.$$

Various versions of this formula appear throughout this thesis, mostly in Chapters 2 and 3, but also formula (5.4.14) is a disguised version. Most often (but not always),

η represents an unstable configuration, and ξ represents the stable configuration that is obtained by only legal topplings. Unless specified otherwise, we always only perform legal topplings.

The abelianness property has been extensively used by Dhar [10]: he defined the addition operator a_x on a configuration as the action of making an addition at site x and stabilizing through topplings, and investigated the group structure of these operators. It appears that the addition operators form an abelian group on so-called *recurrent configurations*. As the name implies, these are configurations that may appear as a result of enough additions (and subsequent stabilization) to any stable configuration. For example, the full configuration with height $2d-1$ at every site is recurrent, but the empty configuration with height 0 at every site is not (except in the case where the grid only consists of 1 site). As is shown in [39], these configurations are also recurrent as according to the standard definition in Markov chain theory.

The interesting aspect of sandpile models is their *stationary state*. In numerical simulations, the model is observed to settle in a state that exhibits power law distributions (for avalanche characteristics), and scale invariance (for height characteristics), which are hallmarks for criticality in statistical mechanics, or interacting particle models with the parameter tuned to the value where a critical phase transition occurs. Bak, Tang and Wiesenfeld proposed the term "self-organized criticality" for systems that evolve dynamically towards a critical stationary state, without tuning of a parameter. Per Bak has written a book entitled "How nature works" [6], in which he expresses his belief that many dynamical aspects of nature, for example the behavior of the earth's crust or biological evolution, are self-organized critical, and that this concept is best studied by means of simple discrete models like the sandpile model.

Analytically, the stationary state of the abelian sandpile model is found to be the uniform distribution on recurrent configurations. For the model in two dimensions, Priezzhev [44] has derived the limit height probabilities, as the grid size tends to infinity, from which the limit stationary expected height per site can be derived. This value turns out to be 2.1247... In any dimension d, this value is between d and $2d-1$. These are just some of the many analytical results for the abelian sandpile model.

1.2 The infinite volume model

Chapters 2 and 3 are on the "self-organized" part of self-organized criticality of the sandpile model. To explain the approach, consider the following reasoning (see also [14]). Suppose that the grid is very large, and the sandpile model starts from the empty configuration. Then it will take many additions before the first toppling occurs, and even more before the avalanches start to have appreciable size. Thus the average height per site, let us call it the density, is at first increasing. It cannot increase indefinitely, in particular it cannot rise above $2d-1$, since every site has to be stable. But even before this value is reached, there will be avalanches in which more than one boundary site topples, so that the density decreases. Thus the density tends to fluctuate around a constant value, which we call the stationary density. As we make the grid size larger, these fluctuations decrease, so that the density tends to be closer to the stationary value. In the limit of infinite grid size, the density is at the stationary value.

Note that the dynamics of the sandpile model is such that the density only increases through additions, so when there are no topplings going on, and only decreases through topplings of boundary sites, so when there are topplings going on. Put slightly differently, we say that the dynamics steer the model towards the transition point between toppling activity and inactivity. Recall that we posed that the sandpile model reaches a critical state without tuning of a parameter. However, it appears that the density is being tuned by the combined effect of additions and avalanches towards the stationary value, which appears to be a critical value for an activity phase transition.

However, we can only speak of a critical value for a parameter in a model that is defined directly in infinite volume, not the infinite volume limit of a finite volume model. Here I informally introduce this infinite volume sandpile model. This model is in the class of interacting particle models, the density is an explicit parameter that can be tuned, and we will be looking for critical density values where a phase transition occurs (this plan has been carried out in dimension 1, where there is indeed a critical density 1, in [40]).

The first contrast with the abelian sandpile model, is that the model is on the infinite grid. Thus, there is no boundary where grains can be lost. The initial configuration is according to a translation invariant measure, but -another contrast with the abelian sandpile- not necessarily stable. The density is defined as the expected height per site. Finally, in the infinite volume model no additions occur, but the model does evolve in time. In this model, the configuration changes in time by topplings of unstable sites.

When we order the topplings in time, we have to decide how. In Chapter 2, there are two choices of toppling order. In one choice, we choose a sequence V_n of growing volumes, with the entire grid as limit volume, and for each n perform only all possible legal topplings inside V_n. In the second choice, we pretend there is an alarm clock at every grid site, and every time it rings, the site topples when it is unstable. The times at which the clocks ring are according to a Poisson process, that is, the intervals between ring times are random, all independent of each other, and all identically distributed according to an exponential distribution. In Chapter 3, we introduce the more general concept of *toppling procedure*, that is a choice of toppling order that obeys certain restrictions. The above two choices are examples of toppling procedures. For every toppling procedure, we prove the following nice properties (Theorem 3.2.13). Let η be an infinite volume initial configuration, according to a translation invariant measure.

- No matter which toppling procedure we choose, it results, depending only on η, either in infinitely many topplings at every site or in finitely many topplings at every site. In the last case, we call η stabilizable.

- When η is stabilizable, then with every toppling procedure, we find the same number of topplings per site, so η has a stable limit configuration that is independent of the choice of toppling procedure.

Intuitively, it is clear that stabilizability is related to the density. If the density is very low, there would not be much to topple. However, if the density is very high, then it does not seem possible to accommodate all the grains in a stable configuration, no matter how many topplings are performed. In between we hope, for appropriate classes of initial measures, to find a phase transition at a critical density, that is, one precise value to separate "low" and "high" density. Even better would be to find that this critical density coincides with the stationary density for the abelian sandpile, in the limit of large grid size; then we can indeed explain how the abelian sandpile model tends to a critical state.

A first demand for this argument to work is that in the infinite volume model, density is conserved. At first glance this seems to be satisfied: since there are no additions or losses at the boundary, the total number of particles must be constant. However, now we have an infinite grid size, and if grains would, for example, tend to move off to infinity, then the density would decrease. The scenario for this is that every site topples in expectation more than its neighbors. Even though this sounds thoroughly counterintuitive because η is translation invariant, it is not immediately excluded because the number of topplings per site could have infinite expectation. In the first half of Chapter 3, culminating in Lemma 3.2.16, we deal with all subtleties and prove that density is indeed conserved in our model.

A naïve guess would be that stabilizability is a function of the density only. However, in Chapter 2 we give (rather artificial) examples of ergodic, translation invariant measures with density down to d which are not stabilizable, and with density up to $2d-1$ which are.

It might still be true that for sufficiently restricted yet interesting (meaning: suitable to make the comparison with the abelian sandpile model) measures, stabilizability is a function of the density only, so that there would be a critical density at which a stabilizability phase transition occurs. We proved that such a critical density is between d and $2d-1$ (Theorem 3.3.2). Note that for $d = 1$, these values coincide, so that for the model in one dimension we found a real phase transition. We prove for $d = 1$ (Theorem 3.3.3) that product measures with the critical density are not stabilizable.

Meanwhile, the topic of phase transitions in interacting particle models is of interest in itself, so it is worthwhile to see if there are other phase transitions in the infinite volume model. One candidate is a percolation phase transition. Intuitively, increasing the density amounts to increasing the number of topplings necessary to stabilize. So we expect that for really low density the topplings are so rare that after stabilizing, we only here and there find small islands -or clusters- of sites that toppled. But if we increase the density, we expect to find larger clusters, and finally they merge into an infinite cluster; this would be a percolation phase transition. If this occurs before we have reached the stabilizability critical density, then there would be two distinct phase transitions in the model; another possibility is that the two critical values are the same. From simulations, it appears that the percolation critical density is d; so far, we managed to prove that it is strictly larger than 0 (Theorem 3.4.1).

1.3 Zhang's sandpile model

In the literature, a host of sandpile variations [37, 4, 12] has been introduced: one can play this game on different grids, or alter the addition and/or toppling rule. Zhang's model, introduced by Zhang [51] in 1989, is one of these variations. It differs in two aspects from the abelian sandpile model: first, additions do not consist of one grain, but of a random amount of sand, distributed uniformly on the interval $[a, b] \subset [0, 1]$, second, a site is unstable when its height is at least 1, and it topples by dividing its entire content in $2d$ parts, giving one part to each neighbor, and itself becoming empty.

Due to these innocent-sounding modifications, the model loses the abelian property, so that a different order of topplings after an addition may result in a different new configuration. Consequently the model was considered analytically intractable for many years, and the article on which Chapter 4 is based, is the first step in this respect. The key observation is that in dimension 1 the model is partly abelian: one can interchange topplings of unstable sites that occur after the same addition and still obtain the same final configuration. The model has been simulated in higher dimensions [51, 25]. It is then necessary to choose an order of topplings. Most commonly, this choice is made like in the abelian sandpile example: all unstable sites topple in parallel, until there are no unstable sites left.

Here is again an example, of Zhang's model in dimension 1. The initial configuration is:

0.8	0.4	0

Now a time step begins by an addition to a random site, of a random amount of sand in the interval, say, $[0, 1]$. In the example, the middle site is chosen, and the amount is 0.8. The result is:

0.8	1.2	0

Because the middle site is now unstable, an avalanche starts:

0.8	1.2	0	$\longrightarrow$	1.4	0	0.6	$\longrightarrow$	0	0.7	0.6

Since the addition amount is random, a stable site could in principle have any height in $[0, 1)$, and the stationary height distribution could be very different from that of the abelian sandpile, where only discrete height values are encountered. Nevertheless, if one simulates the model on a large grid, it appears that the stationary heights tend to concentrate at $2d$ equidistant values. Zhang called these values "quasi-units", and noted that these quasi-units are reminiscent of the abelian sandpile, where for any grid size only $2d$ stable height values are possible. It seems that altering the addition and toppling rules of the abelian sandpile model, which are local rules, does not have that much effect on the global behavior after all, if the grid size is large. We rigorously proved the existence of quasi-units in the case $d = 1$, $a \geq 1/2$ (Theorem 4.5.11).

The proof is rather long, technical and difficult to read. First we prove that for every initial configuration, the model tends to a unique stationary distribution (Theorem 4.5.1), by presenting a successful coupling [48]. We can then choose the empty configuration as initial configuration, which is convenient because it allows to write the height of a certain site at time t as a sum of parts of additions (formula (4.5.13)). We show that typically, as time increases, this sum is composed of more and more parts that are smaller and smaller. If these parts were independent random variables, then the rest of the proof would be straightforward. Predictably, they are not, and that is why the lengthy technical argument is needed.

1.4 The sandpile model as a growth model

Chapter 5 is an atypical chapter in this thesis, since the methods are combinatorial and not probabilistic. However, to put things right, we do use a result from [32], which was derived using probabilistic methods.

On an infinite grid, the initial configuration consists of h grains of sand on every site. Additions are only made to the origin. These additions will spread out through topplings, and we study the shape of the region over which it spreads out. This region obviously grows as the number of additions, n, increases, but the question is: how does it grow? Does it spread evenly in all directions, so that it tends to become spherical, does it tend to some other shape, or to any shape at all? If it tends to some shape, then we call this shape, scaled to unit volume, the limiting shape of the model.

Another aspect is: how fast does it grow? The speed with which the region grows, depends on h. The smaller h is, the more additions it will take to fill up new sites so that they can start toppling. We choose $h \leq 2d - 2$ to make sure that the region stays finite, but there is no objection to choosing arbitrary large negative values for h: just think of a site with a negative height as a site with a hole of depth $|h|$ that needs to be filled up.

Surprisingly, almost nothing was known about the sandpile model as a growth model; see [27]. It had been noted that the sandpile model can give rise to stunningly beautiful patterns (see Figure 5.1 in this thesis), and that someone should work out the obvious relation with the rotor router model. Using sandpile terms, the rotor router model works as follows: a site is unstable if it contains at least one particle. In a toppling, one particle moves to a neighboring site. The neighbors are chosen in cyclic order, so that after $2d$ consecutive topplings, all neighbors received one grain. Indeed: $2d$ consecutive rotor router topplings equal one sandpile toppling.

It turns out that the key to comparing these models is to introduce the parameter h. Furthermore, both models have the abelian property, so that we are free to choose a suitable order of topplings. To illustrate the sort of conclusions one can then draw, consider the rotor router and the sandpile model with the same value of h and n, so that for both models the initial configuration is the same. Suppose that in the sandpile model we perform all possible sandpile topplings, but in the rotor router model, only multiples of $2d$ rotor router topplings. Then both models have done the same, but only the rotor router model is not finished; we did not perform all possible rotor router topplings yet. Therefore, in the rotor router model the sand spreads further, so that the rotor router region includes the sandpile region. Proposition 5.3.1, Theorem 5.4.3 and Theorem 5.4.8 all follow from reasonings like this, and making use of the known result for the rotor router model, that the limiting shape is a sphere [32].

Chapter 2

Organized versus self-organized criticality

Based on "Organized versus self-organized criticality in the abelian sandpile model", by A. Fey and F. Redig [22]; containing some minor corrections and additions.

Abstract We define stabilizability of an infinite volume height configuration and of a probability measure on height configurations. We show that for high enough densities, a probability measure cannot be stabilized. We also show that in some sense the thermodynamic limit of the uniform measures on the recurrent configurations of the abelian sandpile model (ASM) is a maximal element of the set of stabilizable measures. In that sense the self-organized critical behavior of the ASM can be understood in terms of an ordinary transition between stabilizability and non-stabilizability.

2.1 Introduction

Self-organized criticality (SOC) is a concept introduced in [5] to model power-law behavior of avalanche sizes in various natural phenomena such as sand and rice piles, forest fires, etc. The conceptual point of view in [5] is that this kind of criticality is not tuned by parameters such as temperature or magnetic field, as is the case in critical systems of equilibrium statistical mechanics. This point of view has been questioned by several people, see e.g. [2, 14], where it is argued that the choice of the models exhibiting SOC involves an *implicit* tuning of parameters, and hence SOC is an (interesting) example of ordinary criticality. In the case of the abelian sandpile model, e.g. one can say that the choice of the toppling matrix (which governs the dynamics) having mass equal to zero is a fine tuning. Indeed in the massive or dissipative case (where in the bulk grains are lost upon toppling) the avalanche sizes exhibit exponential decay, so in that case there is no criticality.

Similarly, in [40], the authors investigate the relation between the critical density of some parametric model of random walkers with that of the abelian sandpile model, and prove in $d = 1$ that ASM density corresponds exactly to the transition point in the random walkers model. They further conjecture that this is also true in $d \geq 2$. In this paper we want to continue the relation between an ordinary critical phenomenon and the SOC-state of the abelian sandpile model. This is done through the notion of *stabilizability*. A height configuration is called stabilizable if upon stabilizing it in larger and larger volumes, the number of topplings at a fixed site does not diverge. This implies that we can redistribute the mass in *infinite* volume such that after the redistribution, all sites have a height between 0 and $2d - 1$. Similarly a probability measure ν on height configurations is called stabilizable if it concentrates on the set of stabilizable configurations. The conjecture in [40], inspired by [14] is that there exists $\rho_c > 0$ such that (modulo some restrictions on the measure ν) if the ν expected height $\rho < \rho_c$, then ν is stabilizable, if $\rho > 2d - 1$ it is not stabilizable, and for any $\rho \in (\rho_c, 2d - 1)$ there exist measures ν with expected height ρ which are not stabilizable. Moreover, ρ_c is exactly ρ_s, the expected height in the stationary state of the abelian sandpile model, in the thermodynamic limit.

The aim of this paper is to prove the last two items of this conjecture, and to give some more insight in the regime $\rho \in (\rho_c, 2d-1)$. Our paper is organized as follows: in Section 2.2 we define the notion of stabilizability. In Section 2.3 we precisely state the main conjecture of [40] and prove item 3 of it. In Section 2.3 we prove item 2 of the conjecture, and in Section 2.5 disprove by a counterexample item 1. We further show that in some sense the infinite volume limit of the stationary measure of the abelian sandpile model is "maximal stabilizable". Finally in section 2.7 we introduce the concept of metastability, and give a class of examples of probability measures on height configurations having this property.

2.2 Basic definitions

A height configuration is defined as a map $\eta : \mathbb{Z}^d \to \mathbb{N}$, where by convention the minimal height is chosen to be 0. The set of all height configurations is denoted by $\mathcal{X} = \mathbb{N}^{\mathbb{Z}^d}$. In the remainder of this chapter we will only encounter translation invariant measures for height configurations; we denote the set of translation invariant probability measures on the Borel sigma-field of $\mathcal{X}$ as $\mathcal{T}(\mathcal{X})$. A configuration is called *stable* if for all $x \in \mathbb{Z}^d$, $\eta(x) \leq 2d - 1$. The set of stable configurations is denoted by Ω. Similarly, for $V \subset \mathbb{Z}^d$, Ω_V denotes the set of stable configurations $\eta : V \to \{0, \ldots 2d - 1\}$. By η_V, we denote the configuration η restricted to the volume V.

For $V \subset \mathbb{Z}^d$ the abelian sandpile toppling matrix Δ_V is defined by

$$(\Delta_V)_{x,y} = 2d\delta_{x,y} - \mathbf{1}_{x,y\in V,\ |x-y|=1}. \tag{2.2.1}$$

Its inverse is denoted by

$$G_V(x,y) = (\Delta_V^{-1})_{x,y}. \tag{2.2.2}$$

The probabilistic interpretation of G_V is:

$$G_V(x,y) = \frac{1}{2d}\mathbb{E}_x^V \left(\text{number of visits at site } y\right), \tag{2.2.3}$$

where $\mathbb{E}_x^V$ denotes expectation with respect to the simple random walk started at x, and killed upon exiting V.

Definition 2.2.4. *A height configuration $\eta \in \mathcal{X}$ is called stabilizable if for any sequence of nested volumes $V_n \uparrow \mathbb{Z}^d$, there exist $T^\eta_{V_n} \in \mathbb{N}^{V_n}$ such that*

$$\eta_{V_n} - \Delta_{V_n} T^\eta_{V_n} \in \Omega_{V_n}, \tag{2.2.5}$$

and for any $x \in \mathbb{Z}^d$, $T^\eta_{V_n}(x) \to T^\eta(x) < \infty$ as $n \to \infty$.

The set of all stabilizable configurations is denoted by $\mathcal{S}$. It follows immediately from the definition that for $\eta \in \mathcal{S}$, and $T^\eta = \lim_{n\to\infty} T^\eta_{V_n}$,

$$\eta - \Delta T^\eta \in \Omega, \tag{2.2.6}$$

where Δ, defined by $\Delta_{x,y} = 2d\delta_{xy} - \mathbf{1}_{|x-y|=1}$, is the infinite volume toppling matrix.

Definition 2.2.7. *A probability measure ν on height configurations is called stabilizable if $\nu(\mathcal{S}) = 1$.*

It is clear that the set of stabilizable configurations is a translation invariant subset of $\mathcal{X}$. Therefore any stationary and ergodic probability measure μ on $\mathcal{X}$ satisfies $\mu(\mathcal{S}) \in \{0,1\}$.

Remark that in finite volume $V \subset \mathbb{Z}^d$, the abelian sandpile model is "well-defined". This means that for any height configuration $\eta \in \mathbb{N}^V$, the equation

$$\eta_V - \Delta_V T^\eta_V = \xi_V \in \Omega_V, \tag{2.2.8}$$

with unknowns the couple (T^η_V, ξ_V), has at least one solution, namely for $x \in V$, $T^\eta_V(x)$ equals the number of topplings at x needed to stabilize η in V (see e.g. [39]). Notice that the couple (T^η_V, ξ_V) is not unique, but if T^η_V is such that ξ_V is stable and (2.2.8) holds, then $T^\eta_V(x)$ is *at least* the number of topplings at x needed to stabilize η_V. So the vector collecting the number of topplings (needed to stabilize η_V) is the *minimal* solution T^η_V of the equation (2.2.8). We will always choose this solution in the sequel.

On $\mathcal{X}$ we have the natural pointwise ordering $\eta \leq \xi$ if $\forall x \in \mathbb{Z}^d$, $\eta(x) \leq \xi(x)$. A function $f : \mathcal{X} \to \mathbb{R}$ is called monotone if $\eta \leq \xi$ implies $f(\eta) \leq f(\xi)$. Two probability measures μ, ν on $\mathcal{X}$ are ordered $\mu \leq \nu$ if for any monotone function, $\int f d\mu \leq \int f d\nu$, that is, ν stochastically dominates μ. This is equivalent to the existence of a coupling $\mathbb{P}$ of μ and ν such that $\mathbb{P}(\{(\eta, \xi) : \eta \leq \xi\}) = 1$.

By abelianness, $T^{\eta}_{V_n}(x)$ is non-decreasing in n. Therefore a configuration η is not stabilizable if and only if there exists $x \in \mathbb{Z}^d$ such that $T^{\eta}_{V_n}(x) \uparrow \infty$. By abelianness, the T^{η}_{V} are also monotone functions (in the sense just mentioned) of the configuration η.

Therefore we have the following immediate properties of the set of stabilizable configurations

Proposition 2.2.9. *a) $\mathcal{S}$ is a translation invariant measurable set.*

b) If $\eta \in \mathcal{S}$ and $\xi \leq \eta$, then $\xi \in \mathcal{S}$.

c) If μ is a stabilizable probability measure, and $\nu \leq \mu$, then ν is a stabilizable probability measure.

For translation invariant measures on $\mathcal{X}$, we introduce the density ρ as the expected height of the origin.

Definition 2.2.10. *For a measure $\nu \in \mathcal{T}(\mathcal{X})$, we define the density as*

$$\rho = \mathbb{E}_{\nu}\eta(0),$$

where $\mathbb{E}_{\mu}$ denotes expectation w.r.t. to ν.

We then define the following "critical densities":

Lemma 2.2.11. *Define*

$$\begin{aligned} \rho_c^+ &= \inf\{\rho \geq 0 : \exists \nu \in \mathcal{T}(\mathcal{X}) \textit{ with density } \rho, \ \nu \textit{ is not stabilizable}\}, \\ \rho_c^- &= \sup\{\rho \geq 0 : \forall \nu \in \mathcal{T}(\mathcal{X}) \textit{ with density } \rho, \ \nu \textit{ is stabilizable}\}. \end{aligned} \tag{2.2.12}$$

Then $\rho_c^+ = \rho_c^-$.

Proof. It suffices to see that the set

$$S = \{\rho \geq 0 : \text{ such that } \forall \nu \in \mathcal{T}(\mathcal{X}) \text{ with density } \rho, \ \nu \text{ is stabilizable}\} \tag{2.2.13}$$

is an interval. Suppose that $\rho \in S$ and $\rho' < \rho$. Consider a measure $\nu' \in \mathcal{T}(\mathcal{X})$ with density ρ'. Then there exists a measure $\nu \in \mathcal{T}(\mathcal{X})$ with density ρ and such that $\nu \geq \nu'$. Since ν is stabilizable, by the monotonicity property 2.2.9 item 3, ν' is stabilizable. $\square$

We now introduce the stationary state of the sandpile model, and its thermodynamic limit. Define a configuration allowed in a volume V if for any subset $W \subset V$, the inequality

$$\eta(x) \leq |\{y \in W, |y-x| = 1\}| \tag{2.2.14}$$

is violated for at least one $x \in W$. The set of allowed configurations in volume V is denoted by $\mathcal{R}_V$. It is well known that the set of recurrent configurations of the abelian sandpile model in finite volume V coincides with the set of allowed configurations $\mathcal{R}_V$, and the stationary measure is the uniform measure on $\mathcal{R}_V$. (see e.g. [39] or the basic reference [10]). We denote this measure by μ_V. Recently, it has been proved in [3, 36] that the weak limit $\mu = \lim_{V \uparrow \mathbb{Z}^d} \mu_V$ exists and defines a measure on infinite volume height configurations. Moreover, its support $\mathcal{R}$ is the set of those configurations η such that all restrictions to finite volumes V have the property $\eta_V \in \mathcal{R}_V$. We will call this measure μ the uniform measure on recurrent configurations (UMRC). We will always use the symbol μ for the UMRC and denote its density by ρ_s. In [3] it is proved that μ is translation invariant, while in [26] it is proved that μ is tail-trivial so in particular ergodic under spatial translations.

2.3 Main conjecture and results

In the rest of the paper we will prove point 2 and 3 of the following conjecture appearing in [40], cf. also [14], and we will also give some additional new results and examples.

Conjecture: Let ν be an ergodic probability measure in $\mathcal{T}(\mathcal{X})$.

1. For $\rho < \rho_s$, ν is stabilizable,
2. For $\rho_s < \rho \leq 2d-1$ there exist ν which are not stabilizable,
3. For $\rho > 2d-1$, ν is not stabilizable.

The following theorem settles point 3 of the conjecture.

Theorem 2.3.1. *Suppose that η is according to $\nu \in \mathcal{T}(\mathcal{X})$ with $\rho > 2d-1$. Then η is almost surely not stabilizable.*

Proof. Let G_V be the Green function introduced in (2.2.2). Simple random walk killed upon exiting V will be denoted by $\{X_i, i \in \mathbb{N}\}$, and corresponding expectation by $\mathbb{E}^V_{rw}$. Finally, let τ_V denote the lifetime of this walk. The infinite volume random walk expectation is denoted by $\mathbb{E}_{rw}$. Of course, $\mathbb{E}_{rw}$ and $\mathbb{E}^V_{rw}$-expectation of events before τ_V coincide.

Suppose that η drawn from ν is stabilizable. Then we have

$$\begin{aligned} T^{\eta}_{V_n}(0) &= \sum_{x\in V} G_V(0,x)(\eta(x)-\xi(x)) \\ &= (2d)^{-1}\mathbb{E}_{rw}\left(\sum_{i=0}^{\tau_V}(\eta(X_i)-\xi(X_i))\right), \end{aligned} \tag{2.3.2}$$

and $T^{\eta}_{V_n}(0) \uparrow T^{\eta}(0) < \infty$ as $n \to \infty$. Since the random field η is translation invariant and ergodic, we have, by Proposition 2.8.1,

$$\lim_{n\to\infty} \frac{1}{\tau_{V_n}} \sum_{i=0}^{\tau_{V_n}} \eta(X_i) = \rho$$

$\mathbb{P}_0 \times \nu$ almost surely, where $\mathbb{P}_0$ denotes the path-space measure of the simple random walk starting at 0. For ξ we cannot conclude such a strong statement but we have, by stability

$$\limsup_{n\to\infty} \frac{1}{\tau_{V_n}} \sum_{i=0}^{\tau_{V_n}} \xi(X_i) \le 2d-1.$$

Therefore since $\rho > 2d-1+\delta$ for some $\delta > 0$,

$$\liminf_{n\to\infty} \frac{1}{\tau_{V_n}} \sum_{i=0}^{\tau_{V_n}} (\eta(X_i)-\xi(X_i)) > \delta. \tag{2.3.3}$$

This implies, using Fatou's lemma, and the fact $\tau_{V_n} \to \infty$ as $n \to \infty$, that for any $A > 0$

$$\begin{aligned} & 2d \liminf_{n\to\infty} T^{\eta}_{V_n}(0) = \liminf_{n\to\infty} \mathbb{E}_{rw}\left(\sum_{i=0}^{\tau_{V_n}}(\eta(X_i)-\xi(X_i)\right) \\ \ge\ & \liminf_{n\to\infty} \mathbb{E}_{rw}\left(A\mathbf{1}_{\tau_{V_n}>A} \frac{1}{\tau_{V_n}}\left(\sum_{i=0}^{\tau_{V_n}}(\eta(X_i)-\xi(X_i))\right)\right) \\ \ge\ & \mathbb{E}_{rw}\left(\liminf_{n\to\infty} A\mathbf{1}_{\tau_{V_n}>A} \liminf_{n\to\infty} \frac{1}{\tau_{V_n}}\left(\sum_{i=0}^{\tau_{V_n}}(\eta(X_i)-\xi(X_i))\right)\right) \\ \ge\ & A\delta. \end{aligned} \tag{2.3.4}$$

Since $A > 0$ is arbitrary, we arrive at a contradiction. □

2.4 Adding to the stationary measure

In this section we settle point 2 of the conjecture.

The UMRC μ is obtained as a limit of finite volume stationary measure μ_V. These μ_V are in turn obtained by running the finite volume addition and relaxation process for a long time. Therefore, one can believe that μ is "on the edge" of stabilizability. More precisely if one could still "add mass" to μ, then μ would not be stationary.

However, it is not true that μ is a maximal stabilizable measure in the sense of the FKG ordering of measures. Indeed, one can create the following translation invariant ν: pick a configuration according to μ and flip all the height ones to height four. This measure is strictly dominating μ in FKG sense, but it concentrates on stable configurations. In the last section of this paper we will show that such "artificially stable" measures are in some sense "metastable".

The idea of formalizing the maximality of μ is that "one cannot add mass to μ". For ν_1 and ν_2 probability measures on $\mathcal{X}$, we denote by $\nu_1 \oplus \nu_2$ the distribution of $\eta+\xi$ where η is distributed according to ν_1 and ξ is *independent* of η and distributed according to ν_2.

Definition 2.4.1. *A probability measure ν on $\mathcal{X}$ is called maximal stabilizable if for any ergodic translation invariant ν' with positive density, $\nu \oplus \nu'$ is not stabilizable.*

In order to state our main result of this section, we need some more conditions on the UMRC μ. For a configuration drawn from μ, we define the addition operator $a_{x,V}$ as follows: upon application of $a_{x,V}$ to a configuration, the configuration changes as if we added at x and stabilized only in V. Outside the outer boundary of V, the configuration remains unaltered.

We say that the infinite volume addition operator $a_x = \lim_{V\uparrow\mathbb{Z}^d} a_{x,V}$ is well-defined w.r.t. the UMRC if for μ almost every η, the limit $\lim_{V\uparrow\mathbb{Z}^d} a_{x,V}(\eta)$ exists. We now can state our conditions

Definition 2.4.2. *The UMRC is called canonical if*

1. *The infinite volume addition operators a_x are well-defined w.r.t. the UMRC for any $x \in \mathbb{Z}^d$.*

2. *The UMRC is stationary w.r.t. the action of a_x, i.e., if η is distributed according to the UMRC, then so is $a_x\eta$.*

In [26] we prove that these conditions are satisfied on $\mathbb{Z}^d$, $d \geq 3$. The restriction $d \geq 3$ is however of a technical nature, and we strongly believe that these conditions are satisfied in $d = 2$ also.

The case $d = 1$ has been treated in [40]. They proved the following:

Theorem 2.4.3. *A translation invariant and ergodic measure ν on $\mathbb{N}^{\mathbb{Z}}$ such that $\rho < 1$ is stabilizable. If on the contrary $\rho > 1$, then ν is not stabilizable.*

Remark 2.4.4. For $\nu(\eta(0)) = 1$ one can have both stabilizability and non-stabilizability: e.g. the configuration 202020202... and its shift 020202020... are not stabilizable.

If the UMRC μ is canonical, then one can easily see that finite products of addition operators are well-defined μ a.s. and leave μ invariant. See [26] for a complete proof.

Our main result in this section is the following.

Theorem 2.4.5. *If the UMRC is canonical, then it is maximal stabilizable.*

Proof. We have to prove that $\mu \oplus \nu$ is not stabilizable for any ν translation invariant and ergodic such that $\rho > 0$. A configuration drawn from $\mu \oplus \nu$ is of the form $\eta + \alpha$, where η is distributed according to μ and α independently according to ν.

Suppose $\eta + \alpha$ can be stabilized, then we can write

$$\eta_{V_n} + \alpha_{V_n} - \Delta_{V_n} T^{\eta+\alpha}_{V_n} = \xi_{V_n} \in \Omega_{V_n}, \tag{2.4.6}$$

with $T^{\eta+\alpha}_{V_n} \uparrow T^{\eta+\alpha} < \infty$ as $n \to \infty$. We define $T^{\eta+\alpha,V} \in \mathbb{N}^{\mathbb{Z}^d}$ by

$$\eta + \alpha_V - \Delta T^{\eta+\alpha,V} = \xi^{2,V} \in \Omega. \tag{2.4.7}$$

In words this means that we add according to α only in the finite volume V but we *stabilize in infinite volume*. The fact that $T^{\eta+\alpha,V}$ is finite follows from the fact that the addition operators a_x and finite products of these are well-defined in infinite volume on μ almost every configuration. Since for $W \supset V$

$$\alpha_V \le \alpha_W, \tag{2.4.8}$$

and $T^{\eta+\alpha}_{V_n}$ does not diverge, it is clear that $T^{\eta+\alpha,V}$ is well-defined, by approximating the equation (2.4.7) in growing volumes. Moreover, for $\Lambda \subset \mathbb{Z}^d$ fixed, it is also clear that $(T^{\eta+\alpha,V})_\Lambda$ and $(T^{\eta+\alpha}_V)_\Lambda$ will coincide for $V \supset \Lambda$ large enough. Otherwise, the stabilization of $\eta_{V_n} + \alpha_{V_n}$ would require additional topplings in Λ for infinitely many V_n's, which clearly contradicts that $T^{\eta+\alpha}_{V_n}$ converges (and hence remains bounded). But then we have that for n large enough, $(\xi_{V_n})_\Lambda$ and ξ^{2,V_n}_Λ coincide. For any V, the distribution of $\xi^{2,V}$ is μ, because μ is stationary under the infinite volume addition operators. Therefore, we conclude that the limit $\lim_{n\to\infty} \xi_{V_n} = \lim_{n\to\infty} \xi^{2,V_n} = \xi$ is distributed according to μ. Hence, passing to the limit $V \uparrow \mathbb{Z}^d$ in (2.4.7) we obtain

$$\eta + \alpha - \Delta T^{\eta+\alpha} = \xi, \tag{2.4.9}$$

where η and ξ have the same distribution μ, and where $T^{\eta+\alpha} \in \mathbb{N}^{\mathbb{Z}^d}$. Let $\{X_i, i \in \mathbb{N}\}$ be simple random walk starting at the origin. Then for any function $f : \mathbb{Z}^d \to \mathbb{R}$,

$$M_m = f(X_m) - f(0) - \frac{1}{2d} \sum_{i=0}^{m-1} (-\Delta) f(X_i) \tag{2.4.10}$$

is a mean zero martingale. Applying this identity with $f(x) = T^{\eta+\alpha}(x)$, using (2.4.9) gives that

$$M_m = T^{\eta+\alpha}(X_m) - T^{\eta+\alpha}(0) + \sum_{i=0}^{m-1} (\eta(X_i) + \alpha(X_i) - \xi(X_i)) \tag{2.4.11}$$

is a mean zero martingale (w.r.t. the filtration $\mathcal{F}_m = \sigma(X_r : 0 \le r \le m)$, so η and ξ are *fixed* here).

This gives, upon taking expectation over the random walk

$$\frac{1}{m}\left(\mathbb{E}_{rw} T^{\eta+\alpha}(X_m) - T^{\eta+\alpha}(0)\right) = \frac{1}{m}\mathbb{E}_{rw}\left(\sum_{i=0}^{m-1} (\xi(X_i) - \eta(X_i) - \rho(X_i))\right). \tag{2.4.12}$$

Using now that $T^{\eta+\alpha}(0) < \infty$ by assumption, the ergodicity of μ and ν, and the fact that both ξ and η have distribution μ, we obtain from (2.4.12) upon taking the limit $m \to \infty$ that

$$\begin{aligned} 0 &\le \lim_{m\to\infty} \frac{1}{m}\left(\mathbb{E}_{rw}(T^{\eta+\alpha}(X_m)) - T^{\eta+\alpha}(0)\right) \\ &= \lim_{m\to\infty} \mathbb{E}_{rw}\left(\frac{1}{m}\sum_{i=0}^{m-1} (\xi(X_i) - \eta(X_i) - \alpha(X_i))\right) = -\rho, \end{aligned} \tag{2.4.13}$$

which is a contradiction. □

2.5 Constructive example

In this section we present an explicit example of an addition which leads to infinitely many topplings at the origin in the limit $V \uparrow \mathbb{Z}^d$. It settles point 2 of the conjecture, even in the case when the UMRC is not canonical (in particular for $d \le 4$), and shows that point 1 is not true in the generality of translation invariant and ergodic probability measures on height configurations.

Before presenting the example, let us recall the following fact about the abelian sandpile model, see e.g. [24] and the proof of Proposition 2.8.3. Consider the abelian sandpile model in a finite volume $V \subset \mathbb{Z}^d$. Start from a recurrent stable height configuration $\eta : V \to \{0, \ldots, 2d-1\}$. Add on each boundary site $x \in \partial V$ as many grains as there are "lacking neighbors", i.e., the number of grains added at x equals $\lambda_V(x) = |\{y \in \mathbb{Z}^d : |y - x| = 1, y \notin V\}|$. Then, upon stabilization, each site will topple exactly once and the recurrent configuration η will remain unaltered. If V is a rectangle, and a stable height configuration in V is recurrent, then this "special addition" consists of adding two grains to the corner sites and one grain to the other boundary sites. If $\eta : V \to \{0, \ldots, 2d-1\}$ is recurrent in V, and $W \subset V$, then the restriction η_W is recurrent in W. Therefore, by abelianness, if we add to each boundary site x of W *at least* a number grains $\alpha_x^W = \{y \in Z^d : |y-x| = 1, y \notin W\}$, then, upon stabilization, each site in W will topple *at least* once.

We can now present our example in the case $d = 2$; the generalization to $d \neq 2$ is obvious. Let ω, ω' be independent and distributed according to a Bernoulli measure $\mathbb{P}_p$ on $\{0, 1\}^{\mathbb{Z}}$, with $\mathbb{P}_p(\omega(x) = 1) = p$. Consider the following two dimensional random field $\zeta(x, y) = \omega(x) + \omega'(y)$. This is what we are going to add to a recurrent configuration. In words, if for $x \in \mathbb{Z}, \omega(x) = 1$, then we add one grain to each lattice site of the vertical line $\{(x, y), y \in \mathbb{Z}\}$, and if $\omega'(y) = 1$ then we add one grain to each lattice site of the horizontal line $\{(x, y) : x \in \mathbb{Z}\}$. If we add according to ζ, then there are almost surely infinitely many nested rectangles $R_1, \ldots R_n, \ldots$ surrounding the origin with corner sites where we add two grains and other boundary sites where we add at least one grain.

If we add such a configuration ζ to any recurrent configuration η, then we have that the number of topplings at the origin in the finite volume V is at least the number of nested rectangles R_i that are inside V. Indeed, upon addition according to ζ on such a rectangle, every site inside the rectangle will topple at least once.

Therefore the distribution μ_p of $\eta + \zeta$ where η is drawn from the UMRC μ, is not stabilizable. Since we can choose p arbitrarily close to zero, any density $\rho \in (\rho_s, \rho_s + 2)$ can be attained by μ_p.

To show that we can actually get below ρ_s, remember that the fact that the number of topplings inside V is at least the number of rectangles R_i does only depend on the fact that the configuration to which we add is recurrent, and not on the fact that the configuration is chosen from a particular distribution.

Therefore, consider a translation invariant probability measure μ' concentrating on *a subset* $\mathcal{R}'$ of $\mathcal{R}$. Then with the same reasoning, the distribution μ'_p of $\eta + \zeta$ where η is drawn from the μ', is not stabilizable. Consider therefore μ' to be a weak limit point of the uniform measures on *minimal recurrent configurations*, where "minimal" is in the sense of the pointwise ordering of configurations. Then the distribution μ'_p has its density arbitrary close to that of μ'. This number is strictly less than the stationary density ρ_s. Indeed, it is at most 2, because the all two configuration is recurrent, and $\rho_s > 2$, see e.g. [44].

This shows that point 1 of the conjecture cannot hold in that generality.

Combining our results so far with proposition 2.4 from [40], we conclude

Theorem 2.5.1. *Let $\rho_c^+ = \rho_c^-$ be as in lemma 2.2.11. Then*

$$\rho_c^+ = \inf\{\rho : \nu \ \text{ is translation invariant and } \nu(\mathcal{R}) = 1\}. \tag{2.5.2}$$

For any dimension d, $\rho_c^+ \le \rho_s$, and for $d = 2$, $\rho_c^+ \le 2 < \rho_s$.

Remark 2.5.3. We believe that the strict inequality $\rho_c^+ < \rho_s$ holds in *any* dimension $d > 1$, but this is an open problem as far as we know. In $d = 1$ we have $\rho_c^+ = 1 = \rho_s$, but this is an exceptional case where almost all recurrent configurations are minimal recurrent.

2.6 Other notions of stabilizability

2.6.1 Stabilization in infinite volume

Definition 2.6.1. *A configuration $\eta \in \mathcal{X}$ is called weakly stabilizable if there exists $T^\eta \in \mathbb{N}^{\mathbb{Z}^d}$ and $\xi \in \Omega$ such that*

$$\eta - \Delta T^\eta = \xi. \tag{2.6.2}$$

It is clear that if η is stabilizable, then it is weakly stabilizable and we can choose $T^\eta = \lim_{n\to\infty} T^\eta_{V_n}$. However it is not clear whether there exist unstable configurations which can be stabilized weakly, that is, *directly in infinite volume* but which satisfy $\lim_{n\to\infty} T^\eta_{V_n}(0) = \infty$, i.e., the infinite volume toppling numbers are not obtained as the limit of toppling numbers in larger and larger volumes. In the following proposition we prove that a measure ν with $\rho > 2d - 1$ for all $x \in \mathbb{Z}^d$ cannot be weakly stabilized. This means in words that mass cannot be “swept away” to infinity.

The following example shows that the opposite, importing mass from infinity, is not impossible. Consider $f : \mathbb{Z}^2 \to \mathbb{N}$:

$$f(x, y) = x^2 + y^2,$$

then $\Delta f = -4$ and hence, for example,

$$\overline{6} = \overline{2} - \Delta f, \tag{2.6.3}$$

where $\overline{6}$ (resp. $\overline{2}$) denotes the configuration with height 6 (resp. 2) at every site. However,

$$\lim_{n\to\infty} \Delta^{-1}_{V_n}(\overline{6} - \overline{2}) = \infty,$$

so this “infinite volume toppling” cannot be obtained as a limit of finite volume topplings. Notice however that “toppling” according to f is not “legal” in the following sense: we cannot find an order of topplings such that, performed in this order, only unstable sites topple and at the end every site has toppled $x^2 + y^2$ times. The point of (2.6.3) is that the equality

$$\eta = \xi - \Delta T^\eta \tag{2.6.4}$$

for some $T^\eta \in \mathbb{N}^{\mathbb{Z}^d}$ does not imply that the densities of η and ξ are equal. However as we will see later, the equality in (2.6.4) *does imply* that the density of η is *larger or equal* than that of ξ.

In the following proposition we show point 3 of the conjecture for weak stabilizability.

Proposition 2.6.5. *Let ν be translation invariant ergodic such that $\rho > 2d - 1$. Then ν is not weakly stabilizable.*

Proof. Suppose that there exist $T^\eta \in \mathbb{N}^{\mathbb{Z}^d}$ such that

$$\eta - \Delta T^\eta = \xi, \tag{2.6.6}$$

with ξ stable and η a sample from ν. Let X_n be the position of simple random walk starting at the origin at time n. From (2.6.6) it follows that

$$T^\eta(X_n) - T^\eta(0) - \frac{1}{2d}\sum_{k=0}^{n-1}(\xi(X_k) - \eta(X_k)) \tag{2.6.7}$$

is a mean-zero martingale. Therefore taking expectations w.r.t. the random walk

$$\frac{1}{n}\left(\mathbb{E}_{rw}(T^\eta(X_n)) - T^\eta(0)\right) = \frac{1}{n}\mathbb{E}_{rw}\left(\frac{1}{2d}\sum_{k=0}^{n-1}(\xi(X_k) - \eta(X_k))\right). \tag{2.6.8}$$

By stability of ξ

$$\frac{1}{n}\left(\sum_{k=0}^{n-1}(\xi(X_k) - \eta(X_k))\right) \leq 2d - 1 - \frac{1}{n}\sum_{k=0}^{n-1}\eta(X_k). \tag{2.6.9}$$

Therefore, using dominated convergence and $\rho = 2d - 1 + \delta$ for some $\delta > 0$,

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} \frac{2d}{n}\left(\mathbb{E}_{rw}(T^\eta(X_n)) - T^\eta(0)\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}_{rw}\left(\sum_{k=0}^{n-1}(\xi(X_k) - \eta(X_k))\right) \\ &\leq 2d - 1 - \mathbb{E}_{rw}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\eta(X_k)\right) < -\delta, \end{aligned} \tag{2.6.10}$$

which is a contradiction.

□

2.6.2 Activated walkers system and stabilizability at low density

Dickman proposes in [14] the following mechanism of stabilization. Consider a configuration $\eta \in \mathcal{X}$ according to some measure in $\mathcal{T}(\mathcal{X})$. To each $x \in \mathbb{Z}^d$ is associated a Poisson process N_t^x, and for $x \neq y$ these processes are independent. On the event times of N_t^x a site topples if it is unstable (the walkers are "activated"), otherwise nothing happens. This means that after time t the configuration η evolves towards η_t according to a Markov process with generator

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} \mathbf{1}_{\eta(x) > 2d} \left(f(\eta - \Delta_{x,\cdot}) - f(\eta) \right).$$

One says now that the configuration is stabilizable by this process if for any $x \in \mathbb{Z}^d$ the value $\eta_t(x)$ jumps only a finite number of times. One can write the configuration η_t as

$$\eta_t = \eta_0 - \Delta n_t^\eta, \tag{2.6.11}$$

where n_t^η is the vector collecting at each site the number of topplings at x in $[0, t]$. If η is distributed according to a translation invariant probability measure ν on $\mathbb{N}^{\mathbb{Z}^d}$, then so is n_t^η under the joint measure $\nu \times \mathbb{P}$ where $\mathbb{P}$ is the distribution of the Poisson processes $N_t^x, x \in \mathbb{Z}^d$. Moreover $n_t^\eta(x) \leq N_t^x$ by definition and hence this process conserves the density.

Lemma 2.6.12. *A configuration is stabilizable by the process with generator L if and only if it is stabilizable (in the sense of definition 2.2.7).*

Proof. Suppose η is stabilizable by the process with generator L. Consider then the generator

$$L_V f(\eta) = \sum_{x \in V} \mathbf{1}_{\eta(x) > 2d} \left(f(\eta - (\Delta_V)_{x,\cdot}) - f(\eta) \right)$$

corresponding to toppling inside V only, according to the finite volume toppling matrix, on the event times of the Poisson process N_x^t. Call $n_{t,V}^\eta(x)$ the number of updates of x in $[0, t]$. It is easy to see that $n_{t,V}^\eta(x) \uparrow n_t^\eta(x)$ as $V \uparrow \mathbb{Z}^d$, where n_t^η is defined above. By assumption, for any x there exists τ_x^V such that for any $t \geq \tau_x^V$, $n_{t,V}^\eta(x) = n_{\tau_x^V,V}^\eta(x)$. Moreover, $\tau_x^V \uparrow \tau_x$, where τ_x is such that $n_t^\eta(x) = n_{\tau_x}^\eta(x)$ for any $t > \tau_x$. Therefore in definition 2.2.7 we can identify

$$T_V^\eta(x) = n_{\tau_x^V}^\eta(x)$$

and

$$T^\eta(x) = n_{\tau_x}^\eta(x).$$

Suppose that η is stabilizable in the sense of definition 2.2.7. Then, clearly, in finite volume we have the equality

$$T_V^\eta(x) = n^\eta_{\tau_x,V}(x).$$

Since $T_V^\eta \uparrow T^\eta < \infty$ we have

$$\sup_{V\subset\mathbb{Z}^d} n^\eta_{\tau_x,V}(x) < \infty$$

and hence

$$\tau_x = \sup_{V\subset\mathbb{Z}^d} \tau_x^V$$

is finite $\mathbb{P}$-almost surely. Now pick $t > \tau_x$. Then

$$n_t^\eta(x) = \lim_{V\uparrow\mathbb{Z}^d} n^\eta_{t,V}(x) = \lim_{V\uparrow\mathbb{Z}^d} n^\eta_{\tau_x^V,V}(x), \tag{2.6.13}$$

where in the second step we used that the processes with generator L_V converge to the process with generator L weakly on path space. Indeed for any local function

$$\lim_{V\uparrow\mathbb{Z}^d} L_V(f) = L(f).$$

So the convergence of the processes follows from the Trotter-Kurtz theorem. The right hand side of (2.6.13) does not depend on t anymore. Hence η is stabilizable by the process with generator L. □

In [40] the authors prove that there exists $\rho_c' > 0$ such that if ν is an ergodic translation invariant measure with density $\rho < \rho_c'$, then ν is stabilizable by the process with generator L. ρ_c' is the density of minimal recurrent configurations. The following theorem is then an immediate consequence.

Theorem 2.6.14. *There exists $\rho_c' > 0$ such that if ν is any translation invariant ergodic probability measure on $\mathcal{X}$ with $\rho < \rho_c'$, then ν is stabilizable.*

Proof. Combine lemma 2.6.12 with proposition 2.4 from [40]. □

2.7 Metastable measures

Suppose that $\nu \geq \mu$, and ν has a strictly higher density than μ. In that situation ν can still concentrate on stable configurations, and hence be stabilizable. One feels however that such a measure is "on the brink of non-stabilizability". This is formalized in the following definition.

Definition 2.7.1. *A measure ν is called metastable if it is stabilizable and if $\nu \oplus \delta_0$ is not stabilizable with non zero probability, i.e., $\nu \oplus \delta_0(\mathcal{S}) < 1$.*

In words this means that upon stabilizing $\nu \oplus \delta_0$ in volume V, with positive probability the number of topplings $T_V^\eta(0)$ diverges as $V \uparrow \mathbb{Z}^d$. The simplest example of a metastable measure is the measure concentrating on the maximal stable configuration $\overline{2d-1}$. The following theorem shows that there are other non-trivial metastable measures.

Theorem 2.7.2. *Suppose that ν is a translation invariant and ergodic probability measure on Ω, concentrating on the set of recurrent configurations $\mathcal{R}$. Define $I_\eta(x) = \mathbf{1}_{\eta(x)=2d-1}$ and call $\tilde{\nu}$ the distribution of I_η. Suppose that $\tilde{\nu}$ dominates a bernoulli measure $\mathbb{P}_p$ with p sufficiently close to one such that the 1's percolate and the zeros do not percolate. Then ν is metastable.*

Before giving the proof, we need the concept of "wave", see e.g. [24]. Consider a stable height configuration in a finite volume V. Add to x and topple x once (if necessary) and perform all the necessary topplings except a new toppling at x. This is called the first wave. The support of the wave is the set of sites that toppled (it is easy to see that during a wave all sites topple at most once). If x is still unstable, then iterate the same procedure; this gives the second wave etc. For the proof of theorem 2.7.2, we need that the support of a wave is simply connected, i.e., contains no holes. For the sake of completeness, we give a precise definition of this and prove this property of the wave, already formulated in [24].

Definition 2.7.3. *A subset $V \subset \mathbb{Z}^d$ is called simply connected if the set obtained by "filling the squares", i.e., the set*

$$\cup_{i\in V}(i + [-1/2, 1/2]^d)$$

is simply connected.

Lemma 2.7.4. *Let $\eta \in \mathcal{R}_V$ be a recurrent configuration. Then the support of any wave is simply connected.*

Proof. Suppose the support of the first wave contains a maximal (in the sense of inclusion) hole $\varnothing \neq H \subset V$. By hypothesis, all the neighbors of H in V which do not belong to H have toppled once. The toppling of the outer boundary gives an addition to the inner boundary equal to $\lambda_H(x)$ at site x, where $\lambda_H(x)$ is the number of neighbors of x in V, not belonging to H. But addition of this to η_H leads to one toppling at each site $x \in H$, by recurrence of η_H. This is a contradiction because we supposed that $H \neq \varnothing$, and H is not contained in the support of the wave. After the first wave the restriction of the configuration to the volume $V \setminus x$ is recurrent *i.e., recurrent in that volume* $V \setminus x$. Suppose that the second wave contains a hole, then this hole has to be a subset of $V \setminus x$ (because x is contained in the wave by definition), and arguing as before, one sees that the subconfiguration η_H cannot be recurrent. □

We can now give the proof of theorem (2.7.2).

Proof. The idea of the proof is the following. Suppose we have a "sea" of height $2d-1$ and "islands" of other heights, and such that the configuration is recurrent. Suppose the origin belongs to the sea, and we add a grain at the origin. The first wave must be a simply connected subset of $\mathbb{Z}^d$ because the configuration is recurrent. It is clear that the "sea" of height $2d-1$ is part of the wave, and therefore every site is contained in the wave (because if an island is not contained then the wave would not be simply connected). So in the first wave every site topples exactly once, but this implies that the resulting configuration is exactly the same. Hence we have infinitely many waves.

Let us now formalize this. For a given configuration η (distributed according to ν) a volume $V \subset \mathbb{Z}^d$ is called "a lake with islands" if all the boundary sites of V have height $2d-1$, and if from the origin there is a path along sites having height $2d-1$ to the boundary. From the fact that the zeros (corresponding ot the non-full sites) do not percolate, and 1's (corresponding to the full sites) do percolate, it follows that with positive probability, the origin has height $2d-1$ and is in infinitely many nested lakes, i.e. $V_1 \subset V_2, \ldots, V_n \ldots$, with for $i \neq j$, $\partial V_i \cap \partial V_j = \varnothing$. Consider a configuration from that event, consider a volume $V \supseteq V_n$, add a grain at the origin and stabilize the configuration in V. In the first wave all sites will topple once because islands not contained in the wave would contain forbidden subconfigurations, which is impossible since the configuration is recurrent. After the first wave, the only sites that change height are on the boundary of V_n. Therefore, the origin is still unstable and a second wave must start. The sites included in this wave will contain the set of sites included in the first wave needed to stabilize $\eta_{V_{n-1}}$, but this set, with the same argument, is at least V_{n-1}. Continuing like this, one sees that at least n waves are needed for the stabilization of η inside V. Since with positive probability we find infinitely many lakes containing the origin, the number of topplings is diverging in the limit $V \uparrow \mathbb{Z}^d$ with positive probability, which is what we wanted to prove. □

Remark 2.7.5. We believe that every translation invariant probability measure with $\rho \in (\rho_s, 2d-1)$ that concentrates on recurrent configurations is metastable. It is not true that *any* probability measure $\nu \in \mathcal{T}(\mathcal{X})$ with $\rho > \rho_s$ is either metastable or not stabilizable [22], Here is a counterexample.

The counterexample resembles the constructive example from Section 2.5; again we only give the example in two dimensions. Let ω, ω' again be independent and distributed according to a Bernoulli measure $\mathbb{P}_p$ on $\{0,1\}^{\mathbb{Z}}$, with $\mathbb{P}_p(\omega(x) = 1) = p$. Consider the two dimensional random configuration ζ defined by $\zeta(x, y) =$

$3\omega(x)\omega'(y)$. By ρ_p, we denote density of ζ according to this definition. When $p = 1$, ζ is the full configuration. For $p < 1$, ζ is a full configuration with infinitely many horizontal and vertical lines of empty sites. By varying p, choosing $p < 1$, we can choose any density $0 \leq \rho_p < 2d - 1$. Note that for $p < 1$, ζ is not recurrent.

One can prove that if an addition is made to a site x on a configuration such that x is in a rectangle with a non-full boundary, then every site in the rectangle topples a number of times that is at most the distance from x to the boundary, and there are no topplings outside the rectangle. In particular, the total number of topplings is finite. The proof works by ordering the topplings in waves. In Lemma 5.4.2, we give this proof for the special case that the rectangle is a square of full sites and x is the central site. Of course, after these topplings, the boundary may no longer be non-full.

But since for $p < 1$, in ζ every site is a.s. surrounded by infinitely many nested empty rectangular boundaries, and after each addition only one rectangle is affected, we obtain after each addition only a finite number of topplings. Thus, for $p < 1$, ζ is not metastable.

2.8 Appendix

In this appendix we prove the ergodicity of the scenery process, used at several places in this, e.g. in (2.4.13).

Proposition 2.8.1. *Suppose that μ is a translation invariant and ergodic probability measure on $\mathcal{X}$, and $X_n, n \in \mathbb{N}$ is symmetric nearest neighbor random walk on $\mathbb{Z}^d$. Then, if initially η is distributed according to μ, the process $\tau_{X_n}\eta$ defined by*

$$\tau_{X_n}\eta(x) = \eta(x + X_n)$$

is a translation invariant ergodic (in time n) Markov process.

Proof. Let S_d denote the set of unit vectors of $\mathbb{Z}^d$. The process $\tau_{X_n}\eta$ is clearly a Markov process with transition operator

$$Pf(\eta) = \frac{1}{2d}\sum_{e\in S} f(\tau_e\eta).$$

To prove ergodicity (in time n) of the Markov process, we have to show that if f is a bounded measurable function with $Pf = f$, then f is constant μ-almost-surely (see e.g. [46]). So suppose that $Pf = f$, then, by translation invariance of μ,

$$\int f(Pf - f)d\mu = -\frac{1}{4d}\sum_{e\in S_d}\int(\tau_e f - f)^2 d\mu = 0.$$

Hence $\tau_e f = f$, μ almost surely for all unit vectors, and hence $\tau_x f = f$ μ almost surely for all $x \in \mathbb{Z}^d$. By ergodicity of μ (under translations), this implies $f = \int f\mu$ μ-almost surely. $\square$

Finally, we prove a fact about recurrent configurations that we used in our constructive example.

Define two (possibly unstable) height configurations η, ξ on V equivalent if there exists $T_V^\eta : V \to \mathbb{Z}$ such that $\eta - \xi = \Delta_V T_V^\eta$. It is well-known that every equivalence class of this relation contains exactly one recurrent configuration. So if η is a (possibly unstable) height configuration which upon stabilization yields a recurrent configuration ξ, then the relation

$$\eta - \Delta_V T_V^\eta = \xi \tag{2.8.2}$$

simply means that stabilization of η requires $T_V^\eta(x)$ topplings at site $x \in V$ and yields ξ as end result.

Proposition 2.8.3. *Suppose that $V \subset \mathbb{Z}^2$ is a rectangle and η is a recurrent configuration in V. Then upon addition of 2 grains to the corner sites and 1 grain to all other boundary sites, every site will topple once and the resulting configuration remains unaltered.*

Proof. Let $\overline{1}$ denote the column indexed by $x \in V$ of all ones, then $(\Delta_V \overline{1})_x$ equals 4 minus the number of neighbors of x in V. Therefore, the simple identity

$$\eta + \Delta_V \overline{1} - \Delta_V \overline{1} = \eta$$

shows that if η is a recurrent configuration, addition of $(\Delta_V \overline{1})_x$ reproduces η, and makes each site topple once (cf. (2.8.2)). $\square$

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Chapter 3

The infinite volume sandpile model

Reproduction of "Stabilizability and percolation in the infinite volume sandpile model", by A. Fey, R. Meester and F. Redig [21]

abstract We study the sandpile model in infinite volume on $\mathbb{Z}^d$. In particular we are interested in relations between the density ρ of a stationary measure μ on initial configurations, and the question whether or not initial configurations are stabilizable. We prove that stabilizability does not depend on the particular stabilizability rule we adopt. In $d = 1$ and μ a product measure with $\rho = 1$ (the known critical value for stabilizability in $d = 1$) with a positive density of empty sites, we prove that μ is not stabilizable.

Furthermore we study, for values of ρ such that μ is stabilizable, percolation of toppled sites. We find that for $\rho > 0$ small enough, there is a subcritical regime where the distribution of a cluster of toppled sites has an exponential tail, as is the case in the subcritical regime for ordinary percolation.

3.1 Introduction

The sandpile model was originally introduced as a dynamical model to illustrate the concept of self-organized criticality [5]. The model is defined on a finite subset Λ of $\mathbb{Z}^d$, in discrete time. It starts with a stable configuration, that is, every site has a non-negative height of at most $2d - 1$ sand grains. Every discrete time step, an addition of one sand grain is made to a random site. If this site becomes *unstable*, i.e., has at least $2d$ grains, it *topples*, that is, it gives one grain to each neighbor. This may cause other sites to become unstable, and the topplings continue until every site is stable again. The total of all necessary topplings is called an avalanche, so after the avalanche we have reached the new configuration. This is possible in a finite number of topplings because at the boundary of Λ, grains are dissipated. This model is abelian: the obtained configuration is independent of the order of topplings.

This sandpile model is said to exhibit self-organized critical behavior, for the following reasons. As the model evolves in time, it reaches a stationary state that is characterized, in the large-volume limit, by long-range height correlations and power law statistics for avalanche sizes, and thus reminds one of critical behavior in statistical mechanical models. However, the sandpile model evolves naturally towards this critical state, without apparent tuning of any parameters.

This seeming contrast has been discussed in [14, 22, 40]; it is argued that the model definition in fact does involve tuning. Namely, the instantaneousness of topplings, and the vanishing of dissipation as $\Lambda \uparrow \mathbb{Z}^d$, can be viewed as a tuning of the addition and dissipation rate to 0 respectively. This tuning then would ensure that the model evolves towards the critical point of a parametrized, non-dynamical sandpile model, which can informally be described as follows. We start with an initial height configuration on $\mathbb{Z}^d$ (not necessarily stable) according to a translation invariant probability measure with density ρ, which is the expected height, or number of sand grains per site. We keep toppling until there are no more unstable sites. If this is possible with a finite number of topplings per site, then we obtain the final configuration, and the initial configuration is said to be *stabilizable*. This version of the sandpile model was introduced in [14], and mathematically investigated in [22, 40]. Results so far obtained are: for $d = 1$, any translation invariant probability measure with density $\rho < 1$ is stabilizable, any translation invariant probability measure with density $\rho > 1$ is not stabilizable, for $\rho = 1$ there are cases of stabilizability and non-stabilizability, see [40]. For general d, any translation invariant probability measure with density $\rho < d$ is stabilizable, and any translation invariant probability measure with density $\rho > 2d - 1$ is not stabilizable, in between d and $2d - 1$ there are non-stabilizable and stabilizable cases [22].

The present paper continues this investigation and from here on, when we talk about the sandpile model, we mean the version in infinite volume. In Section 3.2 we introduce notation, introduce general toppling procedures and discuss stabilizability issues. In this section, we also prove that if a random initial configuration is stabilizable, then the expected height (density) is conserved by stabilization. In Section 3.3, we define critical values, and investigate the behavior at the critical point in $d = 1$. We find that configurations chosen according to a non-degenerate product measure, are a.s. not stabilizable. In Section 3.4 we investigate phase transitions for the sandpile model from a new viewpoint: we consider, for stabilizable configurations, percolation of the collection of toppled sites. We look for a critical ρ, not necessarily equal to ρ_c mentioned above, such that for all ρ below this value, there is no infinite cluster of toppled sites.

For a general class of initial distributions and ρ small enough, we find a subcritical regime where not only there is a.s. no infinite cluster of toppled sites, but the distribution of the cluster size has an exponential tail. This corresponds to the subcritical regime for ordinary percolation, thus strengthening the idea of a critical phase transition.

3.2 Toppling procedures and stabilizability

Denote by $\mathcal{X} = \mathbb{N}^{\mathbb{Z}^d}$ the set of all height configurations and by $\Omega = \{0, 1, \ldots, 2d-1\}^{\mathbb{Z}^d}$ the set of stable height configurations.

A toppling at site x applied to the configuration $\eta \in \mathcal{X}$ is denoted by $\theta_x(\eta)$ and defined via

$$\theta_x(\eta)(y) = \begin{cases} \eta(y) - 2d & \text{if } y = x, \\ \eta(y) + 1 & \text{if } |y - x| = 1, \\ \eta(y) & \text{otherwise.} \end{cases} \tag{3.2.1}$$

A toppling at site $x \in \mathbb{Z}^d$ is called *legal* for the configuration $\eta \in \mathcal{X}$, if it is applied to an unstable site, that is, if $\eta(x) \geq 2d$.

The above definition of a toppling gives rise to the definition of the *toppling matrix* Δ associated to the sandpile model. This is a matrix indexed by sites $x, y \in \mathbb{Z}^d$, with entries

$$\Delta_{x,y} = 2d\mathbf{1}_{x=y} - \mathbf{1}_{|x-y|=1}.$$

With this definition, and with δ_x defined to be the vector with entry 1 at x and entry 0 in all other positions, we can write

$$\theta_x(\eta) = \eta - \Delta\delta_x.$$

Definition 3.2.2. *A* toppling procedure *is a measurable map (with respect to the usual sigma-algebra's)*

$$T : [0, \infty) \times \mathbb{Z}^d \times \mathcal{X} \to \mathbb{N} \tag{3.2.3}$$

such that for all $\eta \in \mathcal{X}$,

(a) for all $x \in \mathbb{Z}^d$,

$$T(0, x, \eta) = 0.$$

(b) for all $x \in \mathbb{Z}^d$,

$$t \mapsto T(t, x, \eta)$$

is right-continuous and non-decreasing with jumps of size at most one, i.e., for all $t > 0$, $x \in \mathbb{Z}^d$, $\eta \in \mathcal{X}$, we have

$$T(t, x, \eta) - T(t-, x, \eta) \leq 1.$$

(c) *for all $x \in \mathbb{Z}^d$, in every finite time interval, there are almost surely only finitely many jumps at x.*

(d) *T does not contain an "infinite backward chain of topplings", that is, there is no infinite chain of topplings at sites x_i, $i = 1, 2, \ldots$, occurring at times $t_i > t_{i+1} > \ldots$, where for all i, x_{i+1} is a neighbor of x_i.*

Note that condition (d) is only relevant in continuous time. We interpret $T(t, x, \eta)$ as the number of topplings at site x in the time interval $[0, t]$, when T is applied to the initial configuration $\eta \in \mathcal{X}$. The vector of all such numbers at time t is denoted by $T(t, \cdot, \eta)$. We say that for all t such that $T(t-, x, \eta) < T(t, x, \eta)$, site x topples at time t.

If T is a toppling procedure, then for $\eta \in \mathcal{X}$, $t > 0$, we call

$$\Theta_t^\eta(T) = \{x \in \mathbb{Z}^d : T(t, x, \eta) > T(t-, x, \eta)\}$$

the set of sites that topple at time $t > 0$ (for initial configuration η).

Definition 3.2.4. *Let T be a toppling procedure. The configuration η_t at time $t > 0$ associated to T and initial configuration $\eta \in \mathcal{X}$ is defined to be*

$$\eta_t = \eta - \Delta T(t, \cdot, \eta). \tag{3.2.5}$$

Definition 3.2.6. *A toppling procedure T is called* legal *if for all $\eta \in \mathcal{X}$, for all $t > 0$ and for all $x \in \Theta_t^\eta(T)$, $\eta_{t-}(x) \geq 2d$.*

In words, this means that in a legal toppling procedure, only unstable sites are toppled.

Definition 3.2.7. (a) *A toppling procedure T is called* finite *for initial configuration $\eta \in \mathcal{X}$, if for all $x \in \mathbb{Z}^d$,*

$$T(\infty, x, \eta) := \lim_{t\to\infty} T(t, x, \eta) = \sup_{t\geq 0} T(t, x, \eta) \tag{3.2.8}$$

is finite.

(b) *A legal toppling procedure T is called* stabilizing *for initial configuration $\eta \in \mathcal{X}$ if it is finite and if and the limit configuration η_∞, defined by*

$$\eta_\infty = \eta - \Delta T(\infty, \cdot, \eta) \tag{3.2.9}$$

is stable.

A *random toppling procedure* is a random variable with values in the set of toppling procedures. This can also be viewed as a measurable map

$$T : [0, \infty) \times \mathbb{Z}^d \times \mathcal{X} \times \hat{\Omega} \to \mathbb{N}$$

where $\hat{\Omega}$ denotes a probability space, such that for all $\omega \in \hat{\Omega}$ except a set of measure 0, $T(\cdot, \cdot, \cdot, \omega)$ is a toppling procedure.

Definition 3.2.10. *A toppling procedure is called* stationary *if for all t, the distribution of $T(t, \cdot, \eta)$ is translation invariant when we choose η according to a translation invariant probability measure.*

Next we discuss some examples. These examples have in common that for every t, if η_t contains unstable sites, then these sites will topple within finite time almost surely. As a consequence, for every η, these toppling procedures are either stabilizing or infinite. Moreover, if they are infinite, then $T(\infty, x, \eta) = \infty$ for every x. This can be seen as follows: if there is one site x that topples infinitely many times, then the neighbors of x receive infinitely many sand grains. Therefore, these neighbors need to topple infinitely many times, etc.

1. **Markov toppling processes.** These are examples of random stationary toppling procedures and are defined as follows. Each site $x \in \mathbb{Z}^d$ has a Poisson clock (different clocks are independent) with rate one. When the clock at site x rings at time t and in the configuration η_{t-}, x is unstable, then x is toppled. More formally, the configuration η_t of (3.2.5) is evolving according to the Markov process with generator, defined on local functions $f : \mathcal{X} \to \mathbb{R}$ via
$$Lf(\eta) = \sum_x \mathbf{1}_{\eta(x)\geq 2d}(f(\theta_x\eta) - f(\eta)).$$
It is not hard to see that this procedure satisfies all requirements, in particular (d), of Definition 3.2.2.

   It is also possible to adapt the rate at which unstable sites are toppled according to their height. In that case the Markov process becomes
$$L_c f(\eta) = \sum_x \mathbf{1}_{\eta(x)\geq 2d} c(\eta(x))(f(\theta_x\eta) - f(\eta)),$$
where $c : \mathbb{N} \to \mathbb{R}$ has to satisfy certain conditions in order to make the process well-defined.

2. **Toppling in nested volumes.** This is a deterministic, discrete time toppling procedure. Choose a sequence $V_n \subset V_{n+1} \subset \mathbb{Z}^d$ such that $\cup_n V_n = \mathbb{Z}^d$, but all V_n contain finitely many sites. We start toppling all the unstable sites in V_0 until the configuration in V_0 has no unstable sites left, then we do the same with V_1, etc. We put this into the framework of Definition 3.2.2 as follows. At time $t = 1$ we topple all the unstable sites in V_0 once, at time $t = 2$ we topple all the unstable sites in V_0 if there are still unstable sites left after the topplings at time $t = 1$, etc., until at time $t = t(V_0, \eta)$ no unstable sites are left in V_0; we then start toppling at time $t = t(V_0, \eta)$ all unstable sites in V_1, etc. Since the volumes V_n are finite, all $t(V_n, \eta)$ are finite almost surely.

We will use this procedure several times, but for ease of notation we will reparametrize time such that V_n is stabilized at time n instead of at time $t(V_n, \eta)$.

3. **Topplings in parallel.** Topplings in parallel consists simply in toppling at time t all unstable sites of η_{t-1} once. This toppling procedure is discrete time, deterministic and stationary.

4. **Topplings in waves.** This procedure is only used for initial configurations having a single unstable site, say at $x \in \mathbb{Z}^d$. Toppling in waves is defined for the sandpile model on a finite grid as follows [24]: at $t = 1$, we topple x once and subsequently all other sites that become unstable. All these topplings form the first wave. If after these topplings, x is still unstable, then at $t = 2$ we perform the second wave, etc. In each wave, no site topples more than once.

   This does not fit into our framework, because in each wave, all topplings except the toppling at x, are illegal. Nevertheless, we want each wave to be completed in finite time. Therefore, we define topplings in waves as follows: At $t = 1$, we topple site x once. Then for $i = 2, 3, \ldots$, at times $2 - \frac{1}{i}$, we consecutively topple all sites that are unstable except site x. That way, the first wave is completed at time $t = 2$. All other waves proceed similarly. Since in each wave no site topples more than once, this procedure is well-defined.

Definition 3.2.11. *(a) A configuration $\eta \in \mathcal{X}$ is called* stabilizable *if there exists a stabilizing legal toppling procedure.*

(b) A probability measure μ on $(\mathcal{X}, \mathcal{F})$ is called stabilizable if μ-almost every η is stabilizable.

Example 3.2.12. We give an example of a configuration that is not stabilizable: Consider the configuration ξ in $\mathbb{Z}$ where all sites have height 1, except the origin, which has height 2. From trying out by hand it should become clear that this configuration is not stabilizable. We may choose to topple in waves, since there is only one unstable site. In our case, in each wave every site topples exactly once, so that at after each wave we obtain the same configuration. Then there are infinitely many waves. Alternatively, we may choose to topple in parallel. Then in our case, the height of the origin alternates between 0 and 2, so that the origin topples infinitely many times. From the forthcoming Theorem 3.2.13, we can use either (3) or (4) to conclude that ξ is not stabilizable.

In [22], Definition 2.4, stabilizability is defined in terms of toppling in nested volumes, and in [22], Lemma 6.12, it is proved that this definition of stabilizability is equivalent for this toppling procedure and the Markov toppling procedure. Here, we extend this result: we prove that if η is stabilizable, then irrespective of what legal toppling procedure we choose, it will always be finite, and irrespective of what stabilizing procedure we choose, we always obtain the same stable configuration. On the other hand, if we find one infinite legal toppling procedure for η, then we know that η is not stabilizable. This can also be concluded from the existence of a legal toppling procedure in which every site topples at least once.

Let T, T' be two toppling procedures, which are finite for initial configuration η. Then we write

$$T' \preceq_\eta T$$

if for all $x \in \mathbb{Z}^d$

$$T'(\infty, x, \eta) \leq T(\infty, x, \eta).$$

Theorem 3.2.13. *Let T, T' be two legal toppling procedures, which are both finite for initial configuration η,*

1. *If T is stabilizing for η, then*
$$T' \preceq_\eta T. \tag{3.2.14}$$

2. *If T and T' are two stabilizing toppling procedures for η, then for all $x \in \mathbb{Z}^d$*
$$T'(\infty, x, \eta) = T(\infty, x, \eta).$$
In particular, this means that for stabilizable η, the limit configuration η_∞ is well-defined.

3. *For* stabilizable *$\eta \in \mathcal{X}$, there does not exist a non-finite legal toppling procedure.*

4. *If T is stabilizing for η, then there is at least one site x that does not topple, that is, there is at least one site x for which $T(\infty, x, \eta) = 0$.*

Proof. The proof of Statement 1 is inspired by an argument that appears in [13] and [45] in the context of finite grids or discrete time toppling procedures.

For every x, we define a time $\tau_x := \sup\{t : T'(t, x, \eta) \leq T(\infty, x, \eta)\}$, and we call all topplings in T' that occur at times strictly larger than τ_x, "extra" topplings. We suppose the converse of Statement 1, that is, we suppose that there is at least one extra toppling.

Suppose an extra toppling occurs at site y, at time $t_y < \infty$. Then just before time t_y, the number of topplings at site y is at least $T(\infty, y, \eta)$. Moreover, in order for this extra toppling to be legal, site y must be unstable just before time t_y. Thus we find, following T', that

$$\begin{aligned} 2d \le \eta_{t_y-}(y) &= \eta(y) - (\Delta T'(t_y-,\cdot,\eta))(y) \\ &= \eta(y) - 2dT'(t_y-,y,\eta) + \sum_{x\sim y} T'(t_y-,x,\eta) \\ &\le \eta(y) - 2dT(\infty,y,\eta) + \sum_{x\sim y} T'(t_y-,x,\eta), \end{aligned}$$

where the sum $\sum_{x\sim y}$ runs over all neighbors of y. Since T is stabilizing, we have

$$2d > \eta(y) - 2dT(\infty,y,\eta) + \sum_{x\sim y} T(\infty,x,\eta),$$

so for at least one $x \sim y$, $T'(t_y-,x,\eta) > T(\infty,x,\eta)$. In other words, for an extra toppling at site s to be legal, it is necessary that it is preceded by at least one extra toppling at one of its neighbors. Then for this extra toppling, we can make the same observation. Continuing this reasoning, we find that in order for the extra toppling at s to be legal, we need an infinite backward chain of extra topplings, occurring in finite time. But then T' does not satisfy item (d) of Definition 3.2.2. This proves Statement 1.

To prove Statement 2, we simply observe that if T and T' are both stabilizing, then according to the above, $T' \preceq_\eta T$ and $T \preceq_\eta T'$, so that they must be equal.

To prove Statement 3, let T be a stabilizing toppling procedure, and T'' a non-finite legal toppling procedure. Since T'' is non-finite there exists $x \in \mathbb{Z}^d$ such that $T''(t,x,\eta) \uparrow \infty$ as $t \uparrow \infty$. For some $w < \infty$, we define T''_w as follows: for all $t \le w$, $T''_w(t,\cdot,\eta) = T''(t,\cdot,\eta)$, but for all $t > w$, $T''_w(t,\cdot,\eta) = T''(w,\cdot,\eta)$. In words, T''_w performs all topplings according to T'' up to time w, but then stops toppling. T''_w is a finite legal toppling procedure by item (c) of Definition 3.2.2, and hence by Statement 1 of this theorem, $T''_w(\infty,x,\eta) \le T(\infty,x,\eta)$. By letting $w \to \infty$ we obtain a contradiction.

To prove Statement 4, suppose that there is a stabilizing toppling procedure T such that $T(\infty,x,\eta) > 0$ for all x. For every x, we call the toppling that occurs according to T at time $t_x := \min\{t : T(t,x,\eta) = T(\infty,x,\eta)\}$, the "last" toppling. Since T is stabilizing, t_x is finite for all x.

We define $\bar{T}$ as

$$\bar{T}(t,x,\eta) := \min\{T(t,x,\eta), T(\infty,x,\eta) - 1\},$$

so that for all x, $\bar{T}(\infty,x,\eta) = T(\infty,x,\eta) - 1$. In words, $\bar{T}$ contains all topplings according to T except the last one at each site. Note that $\bar{T}$ is a finite, but not a priori legal toppling procedure. However, we have

$$\eta - \Delta\bar{T}(\infty,\cdot,\eta) = \eta - \Delta T(\infty,\cdot,\eta) = \eta_\infty,$$

so that after all topplings according to $\bar{T}$, we have a stable configuration. Now the argument proceeds as in the proof of Statement 1: we have, for some site v,

$$\begin{aligned} 2d \le \eta_{t_v-}(v) &= \eta(v) - 2dT(t_v-, v, \eta) + \sum_{x \sim v} T(t_v-, x, \eta) \\ &= \eta(v) - 2d\bar{T}(\infty, v, \eta) + \sum_{x \sim v} T(t_v-, x, \eta), \end{aligned}$$

whereas

$$2d > \eta(v) - 2d\bar{T}(\infty, v, \eta) + \sum_{x \sim v} \bar{T}(\infty, x, \eta).$$

Similar as in the proof of Statement 1, we conclude that for the last toppling at v to occur legally, it must have been preceded by an infinite backward chain of last topplings, so that T cannot satisfy item (d) of Definition 3.2.2. □

Remark 3.2.15. Note that if μ is stabilizable and ergodic, then the induced measure on limit configurations is also ergodic since it is a factor of μ.

We now prove that a finite legal toppling procedure conserves the density. From here on, we will denote by $\mathbb{E}_\mu$, $\mathbb{P}_\mu$ expectation resp. probability with respect to μ.

Lemma 3.2.16. *Let μ be a translation invariant and ergodic probability measure on $\mathcal{X}$ such that $\mathbb{E}_\mu(\eta(0)) = \rho < \infty$. Suppose furthermore that μ is stabilizable. Then the expected height is conserved by stabilization, that is,*

$$\mathbb{E}_\mu(\eta_\infty(0)) = \rho$$

Proof. Without loss of generality, we assume that the toppling procedure that stabilizes μ is stationary, and is moreover such that for all t, $\mathbb{E}_\mu(T(t, x, \eta)) < \infty$ (we can e.g. choose the Markov toppling procedure, where $T(t, x, \eta)$ is dominated by a Poisson process).

At time t we then have

$$\eta_t(x) = \eta(x) - \sum_y \Delta_{x,y} T(t, y, \eta)$$

which upon integrating over the distribution of η gives

$$\mathbb{E}_\mu(\eta_t(x)) = \mathbb{E}_\mu(\eta(x)) = \rho.$$

Therefore, using Fatou's lemma

$$\rho_\infty := \mathbb{E}_\mu(\eta_\infty(0)) = \mathbb{E}_\mu\left(\lim_{t\to\infty} \eta_t(0)\right) \le \liminf_{t\to\infty} \mathbb{E}_\mu(\eta_t(0)) = \rho.$$

The inequality $\mathbb{E}_\mu(\eta_\infty(0)) \geq \rho$ is proved in [22]; we give a somewhat different argument here. Let X_n denote the position of simple random walk starting at the origin (independent of η), and denote $\mathbb{E}_{rw}, \mathbb{P}_{rw}$ expectation and probability with respect to this random walk. We start by choosing a stabilizable η with limit η_∞, and for a moment we consider this η and η_∞ fixed.

From the relation

$$\eta_\infty(x) = \eta(x) - \sum_y \Delta_{x,y} T(\infty, y, \eta)$$

we obtain

$$\frac{1}{n}\mathbb{E}_{rw}\left(\sum_{k=0}^{n-1}(\eta_\infty(X_k) - \eta(X_k))\right) = \frac{2d}{n}\mathbb{E}_{rw}(T(\infty, X_n, \eta) - T(\infty, 0, \eta)) \tag{3.2.17}$$

By letting $n \to \infty$, this leads to

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}_{rw}\left(\sum_{k=0}^{n-1}\eta_\infty(X_k)\right) \geq \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}_{rw}\left(\sum_{k=0}^{n-1}\eta(X_k)\right). \tag{3.2.18}$$

If we now finally choose η according to μ, which is ergodic, then the limiting measure is also ergodic according to Remark 3.2.15. By ergodicity of the scenery process $\{\eta(X_n) : n \in \mathbb{N}\}$ (see e.g. [22], Proposition 8.1), it follows that for μ-a.e. η, the right hand side is equal to ρ, and the left hand side is equal to ρ_∞. This proves that $\rho_\infty \geq \rho$. □

3.3 Criticality and critical behavior

Let $\mathcal{P}(\mathcal{X})$ denote the set of all translation invariant probability measures on $(\mathcal{X}, \mathcal{F})$. We say that a subset $\mathcal{M}$ of $\mathcal{P}(\mathcal{X})$ is *density complete* if for all $\rho \in [0, \infty)$ there exists $\mu \in \mathcal{M}$ such that $\mu(\eta(0)) = \rho$.

Let $\mathcal{M} \subseteq \mathcal{P}(\mathcal{X})$ be density complete. We define the $\mathcal{M}$-*critical density* for stabilizability to be

$$\rho_c(\mathcal{M}) = \sup\{\rho > 0 : \forall \mu \in \mathcal{M} \text{ with } \mu(\eta(0)) = \rho, \mu \text{ is stabilizable}\}. \tag{3.3.1}$$

Of course, it can be questioned whether the density is the only relevant parameter distinguishing between stabilizability and non-stabilizability. It is certainly the most natural parameter, and is considered in the numerical experiments of [14]. In [22], a related notion of *maximal stabilizability* is introduced.

It is clear that $\rho_c(\mathcal{M}) \leq \rho_c(\mathcal{M}')$ for $\mathcal{M} \supseteq \mathcal{M}'$. Natural choices for $\mathcal{M}$ are a one-parameter family of product measures such as the set of Poisson product measures with parameter ρ, the set of all product measures, or simply $\mathcal{M} = \mathcal{P}(\mathcal{X})$.

The following results are reformulations of results in [40] and [22].

Theorem 3.3.2. *(a) For $\mathcal{M} = \mathcal{P}(\mathcal{X})$, and for all d,*

$$\rho_c(\mathcal{M}) = d.$$

(b) For all $\mathcal{M}$ density complete, we have

$$d \leq \rho_c(\mathcal{M}) \leq 2d - 1.$$

In particular, when $d = 1$ and for all $\mathcal{M}$ density complete, we have

$$\rho_c(\mathcal{M}) = 1.$$

We now specialize to the case $d = 1$. Accordingly, let μ be a one-dimensional product measure with density $\rho = 1$. From Theorem 3.3.2, we know that for $\rho < 1$, μ is stabilizable, and that for $\rho > 1$ it is not. The next result deals with the critical case $\rho = 1$.

Theorem 3.3.3. *Let μ be a one-dimensional product measure with $\rho = 1$ such that $\mu(\eta(0) = 0) > 0$. Then μ is not stabilizable.*

We will start with the proof of part (a), which is a lot more complicated than the proof of part (b). Our strategy for the proof of part (a) will be to show that there a.s. exists a non-finite legal toppling procedure. This implies, using Theorem 3.2.13, that μ is not stabilizable. In order to do so, we will use topplings in nested volumes, but for the proof it will be important to define an intermediate toppling procedure, during which we only stabilize in volumes of the form $[0, n]$, i.e., we increase the stabilized volume only to one side. After stabilization of the interval $[0, n]$, the outer boundary sites -1 and $n+1$ possibly contain, on top of their original height, extra grains that were removed from the interval $[0, n]$ during stabilization; all other sites outside $[-1, n+1]$ still have their original height.

For a while, we concentrate on this one-sided procedure. In this section, we denote by η_n the configuration that results from stabilizing in the interval $[0, n]$. As in the nested volumes toppling procedure, we re-define *time* as to match this notation: stabilization of $[0, n]$ takes place at time n so that η_n is the configuration reached at time n.

We will work with the number and positions of empty sites of η_n in $[0, n]$, and we will call such an empty site "a 0" of η_n. In Figure 3.1 we illustrate the dynamics of the 0's in this procedure. Time (in the new sense) is plotted vertically, and position horizontally. At every time, when you look horizontally, the black dots represent the positions of the 0's at that time in the interval $[0, n]$. In addition, the outer boundary sites of $[0, n]$ are also colored black. The configuration outside the stabilized interval is not shown. Thus, the picture does not give complete information about the configuration η_n, it only shows the positions of the 0's and the width of the stabilized interval. Our strategy is to show that during the one-sided procedure, despite the fact that infinitely often new 0's are created, we infinitely often encounter a configuration that does not contain a 0. Every time this occurs, there is a fixed positive lower bound for the probability that the origin topples. This will then imply that the origin topples infinitely many times a.s.

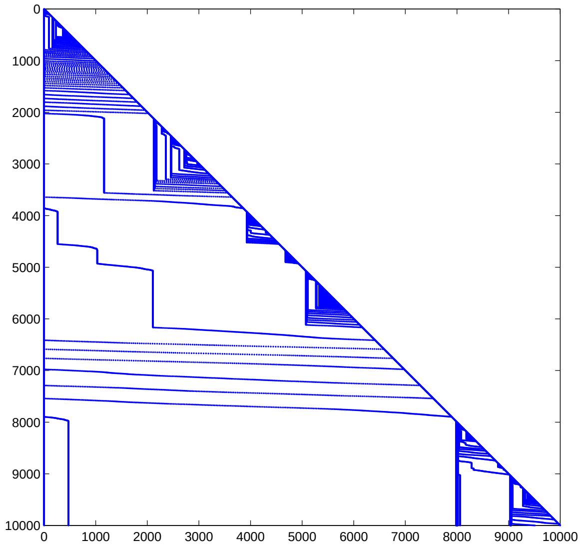

Figure 3.1: The first 10000 time steps for stabilizing η according to Poisson(1) product measure in nested volumes $[0, n]$. The y-axis represents time. A black indicates an empty site (a "0"). In addition, the outer boundary sites of $[0, n]$ are also colored black. See the text for further explanation.

In order to show that infinitely often there are no 0's, we need to analyse the dynamics of the 0's in great detail. We are going to view the 0's as objects that can move, disappear or be created. In order to precisely define these terms, we organize the topplings that occur in time step n into waves. If at time $n-1$ site n is unstable (at time $n-1$, all sites $0, \dots, n-1$ are stable), then in the first wave at time n, we topple site n once and then all other sites in $[0, n]$ that become unstable, except site n again. If after this wave site n is still unstable, then the second wave starts, etc. We will number the waves $k = 0, \dots, K$, and call $\tilde{\eta}_{n-1,k}$ the configuration after the k^{th} wave, so that $\tilde{\eta}_{n-1,0} = \eta_{n-1}$ and $\tilde{\eta}_{n-1,K} = \eta_n$. Depending on the position of the rightmost 0 after wave $k-1$, wave k has the following effect.

1. If the rightmost 0 of $\tilde{\eta}_{n-1,k-1}$ is at site $n-1$, and after wave k site n is not empty, then the number of 0's has decreased by 1;
2. If there are no 0's in $\tilde{\eta}_{n-1,k-1}$, then all sites in $[0,n]$ topple, after which there is a 0 at the origin, and site -1 has gained a grain.
3. In all other cases, if the rightmost 0 is at position $x-1$ (so that x is the leftmost site that topples) then site $x-1$ gains a grain and site x loses one. In addition, site n loses one grain and site $n+1$ gains one.

These observations inspire the following definition.

Definition 3.3.4. *Let at time n, K be the number of waves. If $K > 0$, then let in wave k, x be the leftmost site that topples.*

- *If $x = n$, and after wave k site n is not empty, we say that the 0 at site $n-1$* disappears*.*
- *If $x = 0$, we say that a new 0 is* created at the origin*.*
- *If $x > 0$, and no 0 disappears, we say that the 0 at site $x-1$* moves *to site x.*

If $\eta_{n-1}(n) = 0$ (this implies $K = 0$) we say that a new 0 is created at the right boundary*.*

Since there may be multiple waves in one time step, multiple things can happen to the 0's. However, note that we have the following restrictions: in each wave, only the rightmost 0 can move. For instance, in the example in Figure 3.1, the 0 that is present at position 472 at time 10000, has been in that position for almost 2000 time steps, and we cannot be sure if it will ever move again some future time (actually, as our proof will show, it will a.s.). Furthermore, only when after a previous wave there are no 0's left, can a new 0 be created at the origin. For instance, in the realization in Figure 3.1, this occurs seven times between $n = 6000$ and $n = 8000$.

We stress that according to the above definition, we actually identify certain 0's in different time steps. A look at Figure 3.1 should convince the reader that this is a natural way to view the 0's, even though in order to do so, it is necessary to break the topplings in each time step up into waves to ensure a correct identification. Once a 0 has been created, it exists until it disappears at the right boundary. This may be in the same time step, but it could also require many time steps. During this time, it may move to the right or remain for some periods of time in the same position.

The time intervals between successive instances where the number of 0's is equal to some given number z, are not i.i.d. time intervals. However, we will show in the following lemma that for all $z > 0$, the time intervals from the moment that the number of 0's becomes $z + 1$ until the first return to a value that is at most z, are i.i.d. time intervals, whose distribution does not depend on z. In the proof, we use that for $z > 0$, the number of 0's can only increase from z to $z + 1$ when a new 0 is created at the right boundary. When $z = 0$ we can have that the number of 0's increases because a new 0 is created at the origin, in which case the proof does not apply.

Lemma 3.3.5. *Let $z > 0$. Let $Z(n)$ be the number of 0's after time n. For $i = 0, 1, \ldots$, let $N_0(z) = 0$, $M_i(z) = \min\{n > N_i(z) : Z(n) \leq z\}$, and $N_i(z) = \min\{n > M_{i-1}(z) : Z(n) = z + 1\}$. Then*

1. *The random variables $\Delta_i(z) = M_i(z) - N_i(z)$ are i.i.d., for all $i > 0$.*
2. *The distribution of $\Delta_i(z)$ does not depend on z, and we denote by Δ a random variable with this distribution.*
3. *If $\liminf_{n\to\infty} Z(n) < \infty$ a.s., then $\liminf_{n\to\infty} Z(n) \leq 1$ a.s. and $\mathbb{P}(\Delta = \infty) = 0$.*

Proof. Since at all times $N_i(z)$, a new 0 is created at the right boundary, it must be the case that

$$\eta_{N_i(z)-1}(N_i(z)) = 0.$$

This new 0 will be present until time $M_i(z)$, and during this time, it cannot move to the left. The key observation is that the dynamics of this new 0 depends only on $\eta(j)$, $j \geq N_i(z)$. In particular, conditioned on the creation of the new 0 at time $N_i(z)$, the value of $M_i(z)$ only depends on these random variables. Since

1. the $M_i(z)$ and $N_i(z)$ are all stopping times, and
2. μ is a product measure,

it follows that the $\Delta_i(z)$ are i.i.d. random variables for every fixed z. Furthermore, it follows immediately that the distribution of $\Delta_i(z)$ is also independent of z. This proves the first two parts of the lemma.

We now proceed with Part 3. Suppose that $\liminf_{n\to\infty} Z(n) > 1$ with positive probability, that is, there is a random variable N, finite with positive probability, such that $Z(n) > 1$ for all $n > N$. We denote by $N_e(1)$ the total number of time intervals $\Delta_i(1)$ (during which $Z(n) > 1$). If $N_e(1) < \infty$, then the last interval has infinite length. However, since $\mathbb{P}(\liminf_{n\to\infty} Z(n) > 1) > 0$, it is the case that $\mathbb{P}(N_e(1) < \infty) > 0$. We calculate, using that the $\Delta_i(1)$'s are independent, $\mathbb{P}(N_e(1) < \infty) = \sum_{k=1}^{\infty} \mathbb{P}(N_e(1) = k) = \sum_{k=1}^{\infty} \prod_{i=1}^{k-1} \mathbb{P}(\Delta_i(1) < \infty)\mathbb{P}(\Delta_k(1) = \infty)$. This equals $\sum_{k=1}^{\infty} \mathbb{P}(\Delta < \infty)^{k-1}\mathbb{P}(\Delta = \infty)$, so that we obtain $\mathbb{P}(\Delta = \infty) > 0$.

So far, we showed that $\liminf_{n\to\infty} Z(n) > 1$ with positive probability, implies that $\mathbb{P}(\Delta = \infty) > 0$. Now we show that $\mathbb{P}(\Delta = \infty) > 0$ implies $\liminf_{n\to\infty} Z(n) =$

∞ a.s. Denote by $N_e(z)$ the total number of time intervals $\Delta_i(z)$ (during which $Z(n) > z$), and call $\mathbb{P}(\Delta = \infty) = p$. Similar to the above computation, we calculate $\mathbb{P}(N_e(z) < \infty) = \sum_{k=1}^{\infty} \mathbb{P}(\Delta < \infty)^{k-1}\mathbb{P}(\Delta = \infty) = \sum_{k=0}^{\infty}(1-p)^k p = 1$. Therefore, $\mathbb{P}(N_e(z) < \infty) = 1$ for all $z > 0$, so that $\liminf_{n\to\infty} Z(n) > z$ a.s. for all $z > 0$. It follows that $\liminf_{n\to\infty} Z(n) = \infty$ a.s. □

Proof of Theorem 3.3.3 We choose η according to μ. As mentioned before, we will show that there a.s. exists a non-finite legal toppling procedure. We will use toppling in nested volumes $[-m, m]$. However, in order to compare this procedure with the one-sided procedure introduced above, we will reach η_m from η in the following way: First, we stabilize the interval $[0, m]$. After this step, site -1 received a number $A^+(m)$ of grains, and $[0, m]$ contains a number $Z^+(m)$ of 0's. Then, we stabilize the interval $[-m, -2]$, in the same way as we stabilized in $[0, m]$. After this step, site -1 received another number $A^-(m)$ of grains, and $[-m, -2]$ contains a number $Z^-(m)$ of 0's. Site -1 is now the only possibly unstable site in $[-m, m]$. Finally, we stabilize all of $[-m, m]$. Note that in this description, for every m we obtain η_m starting from η, whereas in the above presented one-sided procedure, we obtained η_n from η_{n-1}. The numbers $A^+(m)$ and $A^-(m)$ are nondecreasing in m. The sequences $(Z^+(m))$ and $(Z^-(m))$ are independent of each other, and also have the same distribution.

The following discussion will repeatedly involve both the one-sided and the nested volume toppling procedure. To make the distinction clear, we will use indices n or N to refer to time steps for the one-sided procedure, and indices m or M to refer to time steps of the nested volume procedure.

First case. We assume that with positive probability, $\liminf_{n\to\infty} Z^{\pm}(m) = \infty$. If both liminfs are actually infinite and we apply the right one-sided procedure to η, then for every $z > 0$ there is a time $N(z)$ such that for all $n > N(z)$, $[0, n]$ contains at least z 0's. This however implies that the leftmost $z-1$ 0's never move again, which in turn implies that from some n on, grains can never reach site -1 again; a similar argument is valid for the left one-sided procedure on the interval $[-n, -2]$. Hence, there is positive probability that both $A^+(m)$ and $A^-(m)$ do not increase anymore eventually, and therefore remain bounded. However, since both $Z^+(m)$ and $Z^-(m)$ tend to infinity, the number of 0's with fixed positions in both the left and right one-sided procedure tends to infinity. This now is incompatible with stabilization, since after toppling site -1 in the end, we should (if stabilization occurs) obtain a stable configuration η_∞ which should be equal to $\bar{1}$, by Lemma 3.2.16. However, there are simply not enough grains at -1 to fill all the 0's that were created by the one-sided procedures.

Second case. We now know that $\liminf_{m\to\infty} Z^{\pm}(m) < \infty$ a.s. By Lemma 3.3.5 part 3, we conclude that a.s. $\liminf_{n\to\infty} Z(n) \leq 1$, and $\mathbb{P}(\Delta = \infty) = 0$. This implies that all 0's, possibly except the leftmost one, will eventually disappear.

Although the proof of Lemma 3.3.5 Part 3 does not work when $z = 0$, we can a fortiori conclude that also the leftmost 0 must eventually disappear. Indeed, since all other 0's eventually disappear, and since the occurrence of this event only depends on the configurations to the right of such a 0, it follows that no matter what the configuration to the right of a certain 0 is, it will always disappear eventually. Clearly, this is then also true for the leftmost 0. (Note though that the leftmost 0 may disappear without $Z(n)$ decreasing; if in a time step where the leftmost 0 disappears also the origin topples, then a new 0 is created at the origin.)

Finally, since clearly infinitely many 0's are created at the right boundary, we conclude that infinitely often the leftmost 0 disappears. Now consider one time instant N' such that the leftmost 0 disappears at time N'. Whether a new 0 is created at this time, depends on the precise value of $\eta_{N'-1}(N')$. Given that this amount is large enough to make the leftmost 0 disappear, we can either have that the origin topples as well, or we can have that the origin does not topple, so that at time N' there are no 0's. In the last case, if $\eta_{N'}(N'+1) \geq 2$, then the origin topples at time $N'+1$. The probability that $\eta_{N'}(N'+1) \geq 2$ is bounded from below by $\mathbb{P}(\eta(N'+1) \geq 2)$. Thus we have that at every time instant where the leftmost 0 disappears, either the origin topples, or it topples with at least a fixed positive probability one time step later. We conclude that during the one-sided procedure the origin topples infinitely often, so that the procedure is non-finite. $\square$

3.4 Sandpile percolation

We call $\mathcal{T}_t$ the set of all sites that have toppled at least once up to (and including) time t, that is, $\mathcal{T}_t = \{x : T_t(x) > 0\}$. Likewise, we introduce the set of nonempty sites at time t, $\mathcal{V}_t = \{x : \eta_t(x) > 0\}$, and finally $\mathcal{W}_t = \mathcal{T}_t \cup \mathcal{V}_t$, the set of sites that have toppled or are nonempty at time t.

For η stabilizable, these sets have a limit, for example $\mathcal{T}_\infty = \lim_{t\to\infty} \mathcal{T}_t$. We decompose the set $\mathcal{T}_\infty$ in clusters $\mathcal{T}_\infty(x)$, where $\mathcal{T}_\infty(x)$ is the largest connected component of $\mathcal{T}_\infty$ containing x. Sandpile percolation is the study of these clusters.

As in classical percolation, one can define critical densities for the existence or absence of infinite clusters and distinguish between a sub- and supercritical regime. In this section, we are interested in the tail of the cluster size distribution

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n)$$

and in the percolation probability

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| = \infty).$$

For the other sets, definitions and notation are similar.

First, we consider sandpile percolation of toppled sites.

Theorem 3.4.1. *Let μ be a translation invariant product measure with finite generating function of the marginal height distribution, that is, with $\mathbb{E}_\mu(e^{t\eta(0)}) < \infty$ for all t, and with density ρ. If $d = 1$, then for all $\rho < 1$, there exists a constant $c_1 = c_1(\rho) > 0$ such that*

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n) \leq e^{-c_1 n}.$$

If $d > 1$, then for all ρ sufficiently small, there exists a constant $c_d = c_d(d,\rho) > 0$ such that

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n) \leq e^{-c_d n}.$$

For the proof, we need Lemma 4.11 from [23]. We cite this lemma below; the proof reveals that η_t restricted to (a subset of) $\mathcal{T}_t$, is recurrent [10], from which the statement follows.

Lemma 3.4.2. *Let Λ be a subset of $\mathcal{T}_t$, for some toppling procedure. Let β_Λ be the number of internal bonds in Λ, that is, bonds with both endpoints in Λ. Then*

$$\sum_{x \in \Lambda} \eta_t(x) \geq \beta_\Lambda.$$

Proof of Theorem 3.4.1. For every d, we have

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n) = \sum_{m=n}^{\infty} \mathbb{P}_\mu(|\mathcal{T}_\infty(0)| = m) + \mathbb{P}_\mu(|\mathcal{T}_\infty(0)| = \infty).$$

We choose to stabilize in nested boxes B_k of radius k. Recall that we reparametrize time so that at time k, the whole box B_k has been stabilized. Then for every k, the maximum size of $\mathcal{T}_k(0)$ is $(2k+1)^d$, so that we can rewrite

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n) = \lim_{k\to\infty} \mathbb{P}_\mu(|\mathcal{T}_k(0)| \geq n) = \lim_{k\to\infty} \sum_{m=n}^{(2k+1)^d} \mathbb{P}_\mu(|\mathcal{T}_k(0)| = m). \quad (3.4.3)$$

We will derive a bound for $\mathbb{P}_\mu(|\mathcal{T}_k(0)| = m)$. We write

$$\mathbb{P}_\mu(|\mathcal{T}_k(0)| = m) = \sum_{\substack{|\mathcal{C}| = m \\ 0 \in \mathcal{C}}} \mathbb{P}_\mu(\mathcal{T}_k(0) = \mathcal{C}).$$

Then, by Lemma 3.4.2, this implies a minimum number of at least $m-1$ sand grains in $\mathcal{C}$ in η_∞. But since no sand can have entered $\mathcal{C}$ during stabilization -

in fact, grains must have left $\mathcal{C}$ - it also implies that $\mathcal{C}$ contains at least m grains at $t = 0$. Since μ is a product measure, this corresponds for $\rho < 1$ to a large deviation of $\frac{1}{m}\sum_{x\in\mathcal{C}}\eta(x)$, and we can bound the corresponding probability by a Chernov bound for sums of independent random variables, that is, there is a constant $\alpha = \alpha(\rho)$ such that

$$\mathbb{P}_\mu(\mathcal{T}_k(0) = \mathcal{C}) \leq e^{-\alpha m}, \tag{3.4.4}$$

where $\lim_{\rho\downarrow 0}\alpha(\rho) = \infty$ because of the assumption on the generating function.

For $d = 1$, the number of clusters of size m containing the origin is m, and for $d > 1$ there is a constant $\alpha' = \alpha'(d)$ such that the number of clusters of size m containing the origin is at most $e^{\alpha' m}$.

Hence, for $d = 1$ we arrive at

$$\sum_{m=n}^{(2k+1)^d}\mathbb{P}_\mu(|\mathcal{T}_k(0)| = m) \leq \sum_{m=n}^{(2k+1)^d}\sum_{\substack{|\mathcal{C}|\,=\,m\\ 0\,\in\,\mathcal{C}}} e^{-\alpha m} \leq \sum_{m=n}^{(2k+1)^d} me^{-\alpha m} \leq e^{-c_1 n},$$

with c_1 positive for all $\rho < 1$.

For $d > 1$, we calculate

$$\sum_{m=n}^{(2k+1)^d}\mathbb{P}_\mu(|\mathcal{T}_k(0)| = m) \leq \sum_{m=n}^{(2k+1)^d}\sum_{\substack{|\mathcal{C}|\,=\,m\\ 0\,\in\,\mathcal{C}}} e^{-\alpha m} \leq \sum_{m=n}^{(2k+1)^d} e^{(\alpha'-\alpha)m} \leq e^{-c_d n},$$

with c_d positive for ρ small enough. Since these outcomes do not depend on k, when inserting this in (3.4.3) we obtain for $d = 1$

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n) \leq e^{-c_1 n},$$

and for $d > 1$

$$\mathbb{P}_\mu(|\mathcal{T}_\infty(0)| \geq n) \leq e^{-c_d n}.$$

□

Remark 3.4.5. In the proof of Theorem 3.4.1, we used that μ is translation invariant, and that we have, for ρ small enough, a large deviation bound for sums like $\sum_{x\in\Lambda}\eta(x)$, with Λ some connected volume in $\mathbb{Z}^d$. There are many more measures that satisfy these requirements, for instance Gibbs measures, or other sufficiently rapidly mixing measures.

The argument to prove exponential tail of the distribution of $|\mathcal{W}_\infty(0)|$ for small ρ, which in turn implies exponential tail of the distribution of $|\mathcal{V}_\infty(0)|$, is similar:

Theorem 3.4.6. *Let μ be a product measure in d dimensions, with finite generating function of the marginal height distribution, and with density ρ. For ρ sufficiently small, there exists a constant $\gamma_d = \gamma_d(d, \rho) > 0$ such that*

$$\mathbb{P}_\mu(|\mathcal{W}_\infty(0)| \geq n) \leq e^{-\gamma_d n}.$$

Proof. As in the proof of Theorem 3.4.1, we stabilize η in nested boxes B_k, and write (see (3.4.3))

$$\begin{aligned}\mathbb{P}_\mu(|\mathcal{W}_\infty(0)| \geq n) &= \lim_{k\to\infty} \mathbb{P}_\mu(|\mathcal{W}_k(0)| \geq n) \\ &= \lim_{k\to\infty} \left(\sum_{m=n}^{\infty} \mathbb{P}_\mu(|\mathcal{W}_k(0)| = m) + \mathbb{P}_\mu(|\mathcal{W}_k(0)| = \infty) \right).\end{aligned}$$

The cluster $W_k(0)$ consists of the following types of sites: sites that have toppled, sites that did not topple but received at least one grain, and sites that did not topple nor received grains but which were nonempty in η. The first two types of sites we can only find in the box B_{k+1}, but the third type we can also find outside this box. Outside the box B_{k+1}, the configuration did not change yet, so restricted to $\mathbb{Z}^d \setminus B_{k+1}$, we just have independent site percolation of nonempty sites. We take ρ so small that the density of nonempty sites is below the critical value for independent site percolation, so that for every k, $|W_k(0)|$ is finite a.s. We write

$$\mathbb{P}_\mu(|\mathcal{W}_\infty(0)| \geq n) = \lim_{k\to\infty} \sum_{m=n}^{\infty} \mathbb{P}_\mu(|\mathcal{W}_k(0)| = m) = \lim_{k\to\infty} \sum_{m=n}^{\infty} \sum_{\substack{|\mathcal{C}| = m \\ 0 \in \mathcal{C}}} \mathbb{P}_\mu(\mathcal{W}_k(0) = \mathcal{C}),$$

and again derive a bound for $\mathbb{P}_\mu(\mathcal{W}_k(0) = \mathcal{C})$ using that there must have been a certain minimal number of sand grains in $\mathcal{C}$ before stabilization. Suppose $\mathcal{W}_k(0) = \mathcal{C}$. If $\mathcal{C}$ contains a cluster of size $m_t \geq 1$ of toppled sites, with $m_b \geq 2d$ boundary sites, then the number of grains in this region of sites - after as well as before toppling - is at least $m_t - 1 + m_b \geq 2d$, so that the density in this region is at least $\frac{2d}{2d+1}$. $\mathcal{C}$ might contain several of these regions, as well as nonempty sites that did not topple nor receive any grains. Thus, we cannot conclude more than that the density in $\mathcal{C}$ before toppling was at least $\frac{2d}{2d+1}$, which for $\rho < \frac{2d}{2d+1}$ corresponds to a large deviation of $\frac{1}{m}\sum_{x\in\mathcal{C}} \eta(x)$.

The rest of the proof proceeds the same as for Theorem 3.4.1. Note that the fact that we now sum m from n to ∞ instead of to $(2k+1)^d$, makes no difference for the outcome. $\square$

Remark 3.4.7. For $d = 1$, the critical density of nonempty sites is 1, but for all finite ρ we have that $\mathbb{P}_\mu(\eta(0) = 1) < 1$. Therefore, Theorem 3.4.6 is valid for $\rho < \frac{2}{3}$.

However, in $d = 1$ it is easy to see, using for instance the last part of Theorem 3.2.13, that for all $\rho < 1$, $|\mathcal{W}_\infty(0)|$ is finite a.s.

Chapter 4

Zhang's sandpile model

Reproduction of "A probabilistic approach to Zhang's sandpile model", by A. Fey, R. Meester, C. Quant and F. Redig [20]

abstract The current literature on sandpile models mainly deals with the abelian sandpile model (ASM) and its variants. We treat a less known - but equally interesting - model, namely Zhang's sandpile. This model differs in two aspects from the ASM. First, additions are not discrete, but random amounts with a uniform distribution on an interval $[a, b]$. Second, if a site topples - which happens if the amount at that site is larger than a threshold value E_c (which is a model parameter), then it divides its entire content in equal amounts among its neighbors. Zhang conjectured that in the infinite volume limit, this model tends to behave like the ASM in the sense that the stationary measure for the system in large volumes tends to be peaked narrowly around a finite set. This belief is supported by simulations, but so far not by analytical investigations.

We study the stationary distribution of this model in one dimension, for several values of a and b. When there is only one site, exact computations are possible. Our main result concerns the limit as the number of sites tends to infinity. We find that the stationary distribution, in the case $a \geq E_c/2$, indeed tends to that of the ASM (up to a scaling factor), in agreement with Zhang's conjecture. For the case $a = 0$, $b = 1$ we provide strong evidence that the stationary expectation tends to $\sqrt{1/2}$.

4.1 Introduction and main results

With the introduction of the sandpile model by Bak, Tang and Wiesenfeld (BTW), the notion of self-organized criticality was introduced, and subsequently applied to several other models such as forest-fire models, and the Bak-Sneppen model for evolution. In turn, these models serve as a paradigm for a variety of natural phenomena in which, empirically, power laws of avalanche characteristics and/or correlations are found, such as the Gutenberg-Richter law for earthquakes. See [49] for a extended overview.

After the work of Dhar [10], the BTW model was later renamed "abelian sandpile model" (ASM), referring to the abelian group structure of addition operators. This abelianness has since served as the main tool of analysis for this model.

A less known variant of the BTW-model has been introduced by Zhang [51], where instead of discrete sand grains, continuous height variables are used. This lattice model is described informally as follows. Consider a finite subset $\Lambda \subset \mathbb{Z}^d$. Initially, every lattice site $i \in \Lambda$ is given an *energy* $0 \leq E_i < E_c$, where E_c is the so called *critical threshold*, and often chosen to be equal to 1. Then, at each discrete time step, one adds a random amount of energy, uniformly distributed on some interval $[a, b] \subset [0, E_c]$, at a randomly chosen lattice site. If the resulting energy at this site is still below the critical value then we have arrived at the new configuration. If not, an *avalanche* is started, in which all unstable sites (that is, sites with energy at least E_c) "topple" in parallel, i.e., give a fraction $1/2d$ of their energy to each neighbor in Λ. As usual in sandpile models, upon toppling of boundary sites, energy is lost. As in the BTW-model, the stabilization of an unstable configuration is performed instantaneously, i.e., one only looks at the final stable result of the random addition.

In his original paper, Zhang observes, based on results of numerical simulation (see also [25]), that for large lattices, the energy variables in the stationary state tend to concentrate around discrete values of energy; he calls this the emergence of energy "quasi-units". Therefore, he argues that in the thermodynamic limit, the stationary dynamics should behave as in the discrete ASM. However, Zhang's model is not abelian (the next configuration depends on the order of topplings in each avalanche; see below), and thus represents a challenge from the analytical point of view. There is no mentioning of this fact in [25, 51], see however [42]; probably they chose the usual parallel order of topplings in simulations.

After its introduction, a model of Zhang's type (the toppling rule is the same as Zhang's, but the addition is a deterministic amount larger than the critical energy) has been studied further in the language of dynamical systems theory in [8]. The stationary distributions found for this model concentrate on fractal sets. Furthermore, in these studies, emergence of self-organized criticality is linked to the behavior of the smallest Lyapounov exponents for large system sizes. From the dynamical systems point of view, Zhang's model is a non-trivial example of an iterated function system, or of a coupled map lattice with strong coupling.

In this paper we rigorously study Zhang's model in dimension $d = 1$ with probabilistic techniques, investigating uniqueness and deriving certain properties of the stationary distribution. Without loss of generality, we take $E_c = 1$ throughout the paper. In Section 4.2 we rigorously define the model for $d = 1$. We show that in the particular case of $d = 1$ and stabilizing after every addition, the topplings are in fact abelian, so that the model can be defined without specifying the order of topplings. In that section, we also include a number of general properties of stationary distributions. For instance, we prove that if the number of sites is finite, then every stationary distribution is absolutely continuous with respect to Lebesgue measure on $(0, 1)$, in contrast with the fractal distributions for the model defined in [8] (where the additions are deterministic).

We then study several specific cases of Zhang's model. For each case, we prove by coupling that the stationary distribution is unique. In Section 4.4, we explicitly compute the stationary distribution for the model on one site, with $a = 0$, by reducing it to the solution of a delay equation [15].

Our main result is in Section 4.5, for the model with $a \geq 1/2$. We show that in the infinite volume limit, every one-site marginal of the stationary distribution concentrates on a non-random value, which is the expectation of the addition distribution (Theorem 4.5.11). This supports Zhang's conjecture that in the infinite volume limit, his model tends to behave like the abelian sandpile. Section 4.5 contains a number of technical results necessary for proving Theorem 4.5.11, but which are also of independent interest. For instance, we construct a coupling of the so-called reduction of Zhang's model to the abelian sandpile model, and we prove that any initial distribution converges exponentially fast to the stationary distribution.

In Section 4.6, we treat the model for $[a, b] = [0, 1]$. We present simulations that indicate the emergence of quasi-units also for this case. However, since in this case there is less correspondence with the abelian sandpile model, we cannot fully prove this. We can prove that the stationary distribution is unique, and we show that if every one-site marginal of the stationary distribution tends to the same value in the infinite volume limit, and in addition if there is a certain amount of asymptotic independence, then this value is $\sqrt{1/2}$. This value is consistent with our own simulations.

4.2 Model definition

We define Zhang's model in dimension one as a discrete-time Markov process with state space

$$\Omega_N := [0, 1)^{\{1,2,\ldots,N\}} \subset [0, \infty)^{\{1,2,\ldots,N\}} := \Xi_N,$$

endowed with the usual sigma-algebra. We write $\eta, \xi \in \Omega_N$, configurations of Zhang's model and η_j for the jth coordinate of η. We interpret η_j as the amount of energy at site j.

By $\mathbb{P}_\eta$, we denote the probability measure on (the usual sigma-algebra on) the path space $\Omega_N^{\mathbb{N}}$ for the process started in η. Likewise we use $\mathbb{P}_\nu$ when the process is started from a probability measure ν on Ω_N, that is, with initial configuration chosen according to ν. The configuration at time t is denoted as $\eta(t)$ and its jth component as $\eta_j(t)$.

We next describe the evolution of the process. Let $0 \leq a < b \leq 1$. At time 0 the process starts in some configuration $\eta \in \Omega_N$. For every $t = 1, 2, \ldots$, the configuration $\eta(t)$ is obtained from $\eta(t-1)$ as follows. At time t, a random amount of energy U_t, uniformly distributed on $[a, b]$, is added to a uniformly chosen site $X_t \in \{1, \ldots, N\}$, hence $P(X_t = j) = 1/N$ for all $j = 1, \ldots, N$. We assume that U_t and X_t are independent of each other and of the past of the process. If, after the addition, the energies at all sites are still smaller than 1, then the resulting configuration is in Ω_N and this is the new configuration of the process.

If however after the addition the energy of site X_t is at least 1 - such a site is called *unstable* - then this site will *topple*, i.e., transfer half of its energy to its left neighbor and the other half to its right neighbor. In case of a toppling of a boundary site, this means that half of the energy disappears. The resulting configuration after one toppling may still not be in Ω_N, because a toppling may give rise to other unstable sites. Toppling continues until all sites have energy smaller than 1 (i.e., until all sites are *stable*). This final result of the addition is the new configuration of the process in Ω_N. The entire sequence of topplings after one addition is called an *avalanche*.

We call the above model the $(N, [a, b])$-model. We use the symbol $\mathcal{T}_x(\xi)$ for the result of toppling of site x in configuration $\xi \in \Xi_N$. We write $\mathcal{A}_{u,x}(\eta)$ for the result of adding an amount u at site x of η, and stabilizing through topplings.

It is not a priori clear that the process described above is well defined. By this we mean that it is not a priori clear that every order in which we perform the various topplings leads to the same final configuration $\eta(t)$. In fact, unlike in the abelian sandpile, topplings according to Zhang's toppling rule are *not* abelian in general. To give an example of non-abelian behavior, let $N = 2$ and $\xi = (1.2, 1.6)$. Then $\mathcal{T}_1(\mathcal{T}_2(\xi)) = \mathcal{T}_1((2, 0)) = (0, 1)$, whereas $\mathcal{T}_2(\mathcal{T}_1(\xi)) = \mathcal{T}_2((0, 2.2)) = (1.1, 0)$.

Despite this non-abelianness of certain topplings, we will now show that in the process defined above, we only encounter avalanches that consist of topplings with the abelian property. When restricted to a certain subset of Ω_N, topplings are abelian, and it turns out that this subset is all we use. (In particular, the example that we just gave cannot occur in our process.)

Proposition 4.2.1. *The $(N, [a, b])$-model is well defined.*

To prove this, we will need the following lemma. First we introduce some notation. We call a site j of a configuration η

$$\begin{array}{lll} \text{empty} & \text{if} & \eta_j = 0, \\ \text{nonempty} & \text{if} & \eta_j \in (0, 1), \\ \text{unstable} & \text{if} & \eta_j \geq 1. \end{array}$$

Lemma 4.2.2. *Let $\tilde{\Omega}_N \subset \Xi_N$ be the set of all (possibly unstable) configurations such that between every pair of unstable sites there is at least one empty site, and such that the energy of any unstable site is strictly smaller than 2.*

During stabilization after an addition to a configuration in Ω_N, only configurations in $\tilde{\Omega}_N$ are encountered.

Proof. We first prove that for every configuration $\tilde{\eta} \in \tilde{\Omega}_N$, the resulting configuration after toppling of one of the unstable sites is still in $\tilde{\Omega}_N$. An unstable site i of $\tilde{\eta}$ in $\tilde{\Omega}_N$ can have either two empty neighbors (first case), two nonempty neighbors (second case) or one nonempty and one empty (third case).

In the first case, toppling of site i cannot create a new unstable site, since $\frac{1}{2}\tilde{\eta}_i < 1$, but i itself becomes empty. Thus, if there were unstable sites to the left and to the right of i, after the toppling there still is an empty site between them.

In the second and third case, the nonempty neighbor(s) of i can become unstable. Suppose the left neighbor $i-1$ becomes unstable. Directly to its right, at i, an empty site is created. To its left, there was either no unstable site, or first an empty site and then somewhere an unstable site. The empty site can not have been site $i-1$ itself, because to have become unstable it must have been nonempty. For the right neighbor the same argument applies. Therefore, the new configuration is still in $\tilde{\Omega}_N$.

Since $\Omega_N \subset \tilde{\Omega}_N$, and since by making an addition to a stable configuration, we arrive in $\tilde{\Omega}_N$, the above argument shows that in the process of stabilization after addition to a stable configuration, only configurations in $\tilde{\Omega}_N$ are encountered. □

Proof of Proposition 4.2.1. By Lemma 4.2.2, we only need to consider configurations in $\tilde{\Omega}_N$. Now we show that, if $\eta \in \tilde{\Omega}_N$ and i and j are unstable sites in η, then

$$\mathcal{T}_i(\mathcal{T}_j(\eta)) = \mathcal{T}_j(\mathcal{T}_i(\eta)). \tag{4.2.3}$$

To prove this, we consider all different possibilities for x. If x is not a neighbor of either i or j, then toppling of i or j does not change η_x, so that (4.2.3) is obvious. If x is equal to i or j, or neighbor to only one of them, then only one of the topplings changes η_x, so that again (4.2.3) is obvious. Finally, if x is a neighbor of both i and j, then, since $\eta \in \tilde{\Omega}_N$, x must be empty before the topplings at i and j. We then have

$$\mathcal{T}_j(\eta)_x = \frac{1}{2}\eta_j,$$

so that

$$\mathcal{T}_i(\mathcal{T}_j(\eta))_x = \frac{1}{2}\eta_j + \frac{1}{2}\eta_i = \mathcal{T}_j(\mathcal{T}_i(\eta))_x$$

Therefore, also in this last case (4.2.3) is true.

Having established that the topplings of two unstable sites commute, it follows that the final stable result after an addition is independent of the order in which we topple, and hence $\mathcal{A}_{u,x}(\eta(t))$ is well-defined; see [39], Section 2.3 for a proof of this latter fact. □

Remark 4.2.4. It will be convenient to order the topplings in so-called *waves* [24]. Suppose the addition of energy at time t takes place at site k and makes this site unstable. In the first wave, we topple site k and then all other sites that become unstable, *but we do not topple site k again.* After this wave only site k can possibly be unstable. If site k is unstable after this first wave, the second wave starts with toppling site k (for the second time) and then all other sites that become unstable, leaving site k alone, until we reach a configuration in which all sites are stable. This is the state of the process at time t. It is easy to see that in each wave, every site can topple at most once.

4.3 Preliminaries and technicalities

In this section, we discuss a number of technical results which are needed in the sequel, and which are also interesting in their own right. The section is subdivided into three subsections, dealing with connections to the abelian sandpile, avalanches, and nonsingularity of the marginals of stationary distributions, respectively.

4.3.1 Comparison with the abelian sandpile model

We start by giving some background on the abelian sandpile model in one dimension. In the abelian sandpile model on a finite set $\Lambda \subset \mathbb{Z}$, the amount of energy added is a nonrandom quantity: each time step one grain of sand is added to a random site. When a site is unstable, i.e., it contains at least two grains, it topples by transferring one grain of sand to each of its two neighbors (at the boundary grains are lost). The abelian addition operator is as follows: add a particle at site x and stabilize by toppling unstable sites, in any order. We denote this operator by $a_x : \{0,1\}^\Lambda \to \{0,1\}^\Lambda$. For toppling of site x in the abelian sandpile model, we use the symbol T_x. Abelian sandpiles have some convenient properties [10]: topplings on different sites commute, addition operators commute, and the stationary measure on finitely many sites is the uniform measure on the set of so-called *recurrent* configurations. Recurrent (or *allowed*) configurations are characterized by the fact that they do not contain a forbidden subconfiguration (FSC). A FSC is defined as a restriction of η to a subset W of Λ, such that η_x is strictly less than the number of neighbors of x in W, for all x. In [39], a proof can be found that a FSC cannot be created by an addition or by a toppling.

In the one-dimensional case on N sites, the abelian sandpile model behaves as follows. Sites are either empty, containing no grains, or full, containing one grain. When an empty site receives a grain, it becomes full, and when a full site receives a grain, it becomes unstable. In the latter case, the configuration changes in the following manner. Suppose the addition site was x. We call the distance to the first site that is empty to the left i. If there is no empty site to the left, then $i-1$ is the distance to the boundary. j is defined similarly, but now to the right. After stabilization, the sites in $\{x-i,\dots,x+j\}\cap\{1,\dots,N\}$ are full, except for a new empty site at $x-i+j$. Only sites in $\{x-i,\dots,x+j\}\cap\{1,\dots,N\}$ have toppled. The number of topplings of each site is at most equal to $\min\{i,j\}$, but is equal to the minimum of its distances to the endsites of the avalanche, if this is less than $\min\{i,j\}$. For example, boundary sites can never topple more than once in an avalanche. These results follow straightforwardly from working out the avalanche. The recurrent configurations are those with at most one empty site; in the one-dimensional case, a connected subset of Λ of more than one site, with empty sites at its boundary, is a FSC. If we have a configuration with exactly one empty site x say, then after the next addition, there is either no empty site (if the addition was at x) or exactly one empty site whose distribution is uniform over all sites except x.

Here is an example of how a non-recurrent state on 11 sites relaxes through topplings. An addition was made to the 7th site; underlined sites are the sites that topple. The topplings are ordered into waves (see Remark 4.2.4). In the example, the second wave starts on the 5th configuration:

$$\begin{aligned}
110111\underline{2}1101 \quad &\to \quad 11011\underline{2}0\underline{2}101 \to 1101\underline{2}020\underline{2}01 \to 110\underline{2}0121011 \to 1110112\underline{1}011 \\
&\to \quad 11101\underline{2}0\underline{2}011 \to 1110\underline{2}020111 \to 111101\underline{2}0111 \to 11110\underline{2}01111 \\
&\to \quad 11111011111.
\end{aligned}$$

To compare Zhang's model to the abelian sandpile, we label the different states of a site $j\in\{1,\dots,N\}$ in $\eta\in\tilde{\Omega}_N$ as follows:

Definition 4.3.1. *Let $\eta\in\tilde{\Omega}_N$. For every $j\in\{1,\dots,N\}$, we say that η_j is*

$$\begin{array}{lll}
\textit{empty } (0) & \textit{if} & \eta_j = 0, \\
\textit{full } (1) & \textit{if} & \eta_j \in [\frac{1}{2}, 1), \\
\textit{unstable } (2) & \textit{if} & \eta_j \geq 1, \\
\textit{anomalous } (a) & \textit{if} & \eta_j \in (0, \frac{1}{2}).
\end{array} \tag{4.3.2}$$

Definition 4.3.3. *The* reduction *of a configuration $\eta\in\tilde{\Omega}_N$ is the configuration denoted by $\mathcal{R}(\eta)\in\{0,1,2,a\}^{\{1,\dots,N\}}$ corresponding to η by Definition (4.3.2). We denote with $\mathcal{R}(\eta_i)$ the reduced value of site i, that is, $\mathcal{R}(\eta_i)=\mathcal{R}(\eta)_i$.*

For general $0 \leq a < b \leq 1$, we have the following result.

Proposition 4.3.4. *For any starting configuration $\eta \in \Omega_N$, there exists a random variable $T \geq 0$ with $P(T < \infty) = 1$ such that for all $t \geq T$, $\eta(t)$ contains at most one empty or anomalous site.*

To prove this proposition, we first introduce FSC's for Zhang's model. We define a FSC in Zhang's model in one dimension as the restriction of η to a subset W of $\{1, \dots, N\}$, in such a way that $2\eta_j$ is strictly less than the number of neighbors of j in W, for all $j \in W$. From here on, we will denote the number of neighbors of j in W by $\deg_W(j)$. To distinguish between the two models, we will from now on call the above *Zhang-FSC*, and the definition given in Section 4.3.1 *abelian-FSC*. From the definition, it follows that in a stable configuration, any restriction to a connected subset of more than one site, with the boundary sites either empty or anomalous, is a Zhang-FSC. Note that according to this definition, a stable configuration without Zhang-FSC's can be equivalently described as a configuration with at most one empty or anomalous site.

Lemma 4.3.5. *A Zhang-FSC cannot be created by an addition in Zhang's model.*

Proof. The proof is similar to the proof of the corresponding fact for abelian-FSC's, which can be found for instance in [39], Section 5. We suppose that $\eta(t)$ does not contain an FSC, and an addition was made at site x. If the addition caused no toppling, then it cannot create a Zhang-FSC, because no site decreased its energy. Suppose therefore that the addition caused a toppling in x. Then for each neighbor y of x

$$\mathcal{T}_x(\eta)_y \geq \eta_y + \frac{1}{2},$$

so that $2\mathcal{T}_x(\eta)_y \geq 2\eta_y + 1$. Also $\mathcal{T}_x(\eta)_x = 0$, and all other sites are unchanged by the toppling.

We will now derive a contradiction. Suppose the toppling created a Zhang-FSC, on a subset which we call W. It is clear that this means that x should be in W, because it is the only site that decreased its energy by the toppling. For all $j \in W$, we should have that $2\mathcal{T}_x(\eta)_j < \deg_W(j)$. This means that for all neighbors y of x in W, we have $2\eta_y < \deg_W(y) - 1$, and for all other $j \in W$ we have $2\eta_j < \deg_W(j)$. From these inequalities it follows that $W \setminus \{x\}$ was already a Zhang-FSC before the toppling, which is not possible, because we supposed that $\eta(t)$ contained no Zhang-FSC.

By the same argument, further topplings cannot create a Zhang-FSC either, and the proof is complete. $\square$

Remark 4.3.6. We have not defined Zhang's model in dimension $d > 1$, because in that case the resulting configuration of stabilization through topplings is not independent of the order of topplings. But since the proof above only discusses the result of one toppling, Lemma 4.3.5 remains valid for any choice of order of topplings. The proof is extended simply by replacing the factor 2 by $2d$.

Proof of Proposition 4.3.4. If η already contains at most one non-full, i.e., empty or anomalous site, then it contains no Zhang-FSC's, and the first part of the proposition follows. Suppose therefore that at some time t, $\eta(t)$ contains $M(t)$ non-full sites, with $1 < M(t) \leq N$. We denote the positions of the non-full sites of $\eta(t)$ by $Y_i(t)$, $i = 1, \ldots, M(t)$, and we will show that $M(t)$ is nonincreasing in t, and decreases to 1 in finite time. Note that for all $1 \leq i < j \leq M(t)$, the restriction of $\eta(t)$ to $\{Y_i(t), Y_i(t) + 1, \ldots, Y_j(t)\}$ is a Zhang-FSC.

At time $t+1$, we have the following two possibilities. Either the addition causes no avalanche, in that case $M(t+1) \leq M(t)$, or it causes an avalanche. We will call the set of sites that change in an avalanche (that is, all sites that topple at least once, together with their neighbors) the *range* of the avalanche. We first show that if the range at time $t + 1$ contains a site $y \in \{Y_i(t), \ldots, Y_{i+1}(t)\}$ for some i, then $M(t + 1) < M(t)$.

Suppose there is such a site. Then, since $\{Y_i(t) + 1, \ldots, Y_{i+1}(t) - 1\}$ contains only full sites, all sites in this subset will topple and after stabilization of this subset, it will not contain a Zhang-FSC by Lemma 4.3.5. In other words, in this subset at most one empty site is created. But since $Y_i(t)$ and $Y_{i+1}(t)$ received energy from a toppling neighbor, they are no longer empty or anomalous. Therefore, $M(t + 1) < M(t)$.

If there is no such site, then the range is either $\{1, \ldots, Y_1(t)\}$, or $\{Y_{M(t)}(t), \ldots, N\}$, and $M(t+1) = M(t)$. With the same reasoning as above, we can conclude that in these cases, $Y_1(t + 1) < Y_1(t)$, resp. $Y_{M(t)}(t + 1) > Y_{M(t)}(t)$.

Thus, $M(t)$ strictly decreases at every time step where an avalanche contains topplings between two non-full sites. As long as there are at least two non-full sites, such an avalanche must occur eventually. We cannot make infinitely many additions without causing topplings, and we cannot infinitely many times cause an avalanche at $x < Y_1(t)$ or $x > Y_{M(t)}(t)$ without decreasing $M(t)$, since after each such an avalanche, these non-full sites "move" closer to the boundary. □

Remark 4.3.7. This proof also shows that a.s. within finite time, each site topples at least once.

In the case that $a \geq 1/2$, we can further specify some characteristics of the model by comparing it to the one-dimensional abelian sandpile discussed above. We define regular configurations as follows:

Definition 4.3.8. *We call a configuration $\eta \in \Omega_N$* regular *if η contains no anomalous sites, and at most one empty site.*

Proposition 4.3.9. *Suppose $a \geq \frac{1}{2}$. Then*

1. *for any initial configuration η, for all $t \geq N(N-1)$, $\eta(t)$ is regular,*

2. *If $\eta(t)$ is regular with no empty site, then $\eta(t+1)$ contains one empty site whose position is uniform over all sites. If $\eta(t)$ is regular with one empty site at x, say, then $\eta(t+1)$ either contains no empty site (if the addition is at x) or one empty site whose distribution is uniform over all sites except x (if the addition is not at x).*

3. *for every stationary distribution μ, and for all $i \in \{1, \dots, N\}$,*

$$\mu(\eta_i = 0) = \frac{1}{N+1}.$$

In words, this proposition states that if $a \geq 1/2$, then every stationary distribution concentrates on regular configurations. Moreover, the stationary probability that a certain site i is empty, does not depend on i. Note that as a consequence, the stationary probability that all sites are full, is also $\frac{1}{N+1}$.

The property mentioned in part 2 of the proposition will be referred to as the empty site being *almost uniform* on $1, \dots, N$, this notion will be used in Section 5.

To prove this proposition, we need the following lemma. In words, it states that if $a \geq 1/2$ and η contains no anomalous sites, then the reduction of Zhang's model (according to Definition 4.3.3) behaves just as the abelian sandpile model.

Lemma 4.3.10. *For all $u \in [\frac{1}{2}, 1)$, for all $\eta \in \Omega_N$ which do not contain anomalous sites, and for all $x \in \{1, \dots, N\}$*

$$\mathcal{R}(\mathcal{A}_{u,x}(\eta)) = a_x(\mathcal{R}(\eta)), \tag{4.3.11}$$

where a_x is the addition operator of the abelian sandpile model. In both avalanches, corresponding sites topple the same number of times.

Proof. Under the conditions of the lemma, site x can be either full or empty. If x is empty, then upon the addition of $u \geq \frac{1}{2}$ it becomes full. No topplings follow, so that in that case we directly have $\mathcal{R}(\mathcal{A}_{u,x}(\eta)) = a_x(\mathcal{R}(\eta))$.

If η is such that site x is full, then upon addition it becomes unstable. We call the configuration after addition, but before any topplings $\tilde{\eta}$, and use that it is in $\tilde{\Omega}_N$ (see Lemma 4.2.2). To check if in that case $\mathcal{R}(\mathcal{A}_{u,x}(\eta)) = a_x(\mathcal{R}(\eta))$, we only need to prove $\mathcal{R}(\mathcal{T}_x(\tilde{\eta})) = T_x(\mathcal{R}(\tilde{\eta}))$, with $\mathcal{R}(\tilde{\eta}_x) = 2$, since we already know that in both models, the final configuration after one addition is independent of the order of topplings.

In $\mathcal{T}_x(\tilde{\eta})$, site x will be empty. This corresponds to the abelian toppling, because site x contained two grains after the addition, and by toppling it gave one to each neighbor. In $\mathcal{T}_x(\tilde{\eta})$, the energy of the neighbors of x is their energy in η, plus at least $\frac{1}{2}$. Thus the neighbors of site x will in $\mathcal{T}_x(\tilde{\eta})$ be full if they were empty, or unstable if they were full. Both correspond to the abelian toppling, where the neighbors of x received one grain. □

Proof of Proposition 4.3.9. To prove part (1), we note that any amount of energy that a site can receive during the process, i.e., either an addition or half the content of an unstable neighbor, is at least 1/2. Thus, anomalous sites can not be created in the process. Anomalous sites can however disappear, either by receiving an addition, or, as we have seen in the proof of Proposition 4.3.4, when they are in the range of an avalanche.

When we make an addition of at least 1/2 to a configuration with more than one non-full site, then either the number of non-full sites strictly decreases, or one of the outer non-full sites moves at least one step closer to the boundary. We note that η contains at most N non-full sites, and the distance to the boundary is at most $N-1$. When finally there is only one non-full site, then in the next time step it must either become full or be in the range of an avalanche. Thus, there is a random time $T' \leq N(N-1)$ such that $\eta(T')$ is regular for the first time, and as anomalous sites cannot be created, by Proposition 4.3.4, $\eta(t)$ is regular for all $t \geq T'$.

For $t \geq T'$, $\eta(t)$ satisfies the condition of Lemma 4.3.10. This means that the evolution of the reduction of Zhang's model coincide s with that of the abelian sandpile model. Parts (2) and (3) then follow from the corresponding properties of the abelian sandpile. □

4.3.2 Avalanches in Zhang's model

We next describe in full detail the effect of an avalanche, started by an addition to a configuration $\eta(t)$ in Zhang's model. Let $\mathcal{C}(t+1)$ be the range of this avalanche. Recall that we defined the range of an avalanche as the set of sites that change their energy at least once in the course of the avalanche (that is, all sites that topple at least once, together with their neighbors). We denote by $\mathcal{T}(t+1)$ the collection of sites that *topple* at least once in the avalanche. Finally, $\mathcal{C}'(t+1) \subset \mathcal{C}(t+1)$ denotes the collection of anomalous sites that change, but do not topple in the avalanche.

During the avalanche, the energies of sites in the range, as well as U_{t+1}, get redistributed through topplings in a rather complicated manner. By decomposing the avalanche into waves (see Remark 4.2.4), we prove the following properties of this redistribution.

Proposition 4.3.12. *Suppose an avalanche is started by an addition at site x to configuration $\eta(t)$. For all sites j in $\mathcal{C}(t+1)$, there exist $F_{ij} = F_{ij}(\eta(t), x, U_{t+1})$ such that we can write*

$$\eta_j(t+1) = \sum_{i \in \mathcal{T}(t+1)} F_{ij}\eta_i(t) + F_{xj}U_{t+1} + \eta_j(t)\mathbf{1}_{j\in\mathcal{C}'(t+1)}, \tag{4.3.13}$$

with

1. $$F_{xj} + \sum_{i\in\mathcal{T}(t+1)} F_{ij} = \mathcal{R}(\eta(t+1)_j); \tag{4.3.14}$$

2. *for all $j \in \mathcal{C}(t+1)$ such that $\eta_j(t+1) \neq 0$,*

$$F_{xj} \geq 2^{-\lceil 3N/2 \rceil};$$

3. *for all $j \in \mathcal{C}(t+1)$ such that $\eta_j(t+1) \neq 0$, $j \geq x$, we have*

$$F_{x,j+1} \leq F_{xj};$$

and similarly, $F_{x,j-1} \leq F_{xj}$ for $j \leq x$.

In words, we can write the new energy of each site in the range of the avalanche at time $t+1$ as a linear combination of energies at time t and the addition U_{t+1}, in such a way that the prefactors sum up to 1 or 0. Furthermore, every site in the range receives a positive fraction of at least $2^{-\lceil 3N/2\rceil}$ of the addition. These received fractions are such that larger fractions are found closer to the addition site. We will need this last property in the proof of Theorem 4.5.11.

Proof of Proposition 4.3.12. We start with part 1. First, we decompose the avalanche started at site x into waves. We index the waves with $k = 1, \ldots, K$, and write out explicitly the configuration after wave k, in terms of the configuration after wave $k-1$. The energy of site i after wave k is denoted by $\tilde{\eta}_{i,k}$; we use the tilde to emphasize that these energies are not really encountered in the process. We define $\tilde{\eta}_{i,0} = \eta_i(t) + U_{t+1}\mathbf{1}_{i=x}$; note that $\tilde{\eta}_{i,K} = \eta_i(t+1)$.

In each wave, all participating sites topple only once by Remark 4.2.4. We call the outermost sites that toppled in wave k, the *endsites* of this wave, and we denote them by M_k and M'_k, with $M_k > M'_k$. For the first wave, this is either a boundary site, or the site next to the empty or anomalous site that stops the wave. Thus, M_1 and M'_1 depend on $\eta(t)$, x and U_{t+1}. For $K > 1$, all further waves are stopped by the empty sites that were created when the endsites of the previous wave toppled, so that for each $k < K$, $M_{k+1} = M_k - 1$ and $M'_{k+1} = M'_k + 1$. In every wave but the last, site x becomes again unstable. Only in the last wave, x is an endsite, so that at most one of its neighbors topples.

In wave k, first site x topples, transferring half its energy, that is, $\frac{1}{2}\tilde{\eta}_{x,k-1}$, to each neighbor. Then, if x is not an endsite (that is, $k < K$), both its neighbors topple, transferring half of their current energy, that is, $\frac{1}{2}\tilde{\eta}_{x\pm1,k-1} + \frac{1}{4}\tilde{\eta}_{x,k-1}$, to their respective neighbors. Site x is then again unstable, but it does not topple again in this wave. Thus, the topplings propagate away from x in both directions, until the endsites are reached. Every toppling site in its turn transfers half its current energy, including the energy received from its toppling neighbor, to both its neighbors. Writing out all topplings leads to the following expression, for all sites $i \geq x$. A similar expression gives the updated energies for the sites with $i < x$. Note that, when $K > 1$, for every $k > 1$, $\tilde{\eta}_{M_k+1,k-1} = 0$. Only when $k = 1$, it can be the case that site $M_1 + 1$ was anomalous, so that $\tilde{\eta}_{M_1+1,0} > 0$.

$$\begin{aligned}
\tilde{\eta}_{x,k} &= \left(\frac{1}{2}\tilde{\eta}_{x+1,k-1} + \frac{1}{4}\tilde{\eta}_{x,k-1}\right)\mathbf{1}_{M_k > x} + \left(\frac{1}{2}\tilde{\eta}_{x-1,k-1} + \frac{1}{4}\tilde{\eta}_{x,k-1}\right)\mathbf{1}_{M'_k < x}, \\
\tilde{\eta}_{i,k} &= \sum_{n=x}^{i+1} \frac{1}{2^{i+2-n}}\tilde{\eta}_{n,k-1}, \qquad \text{for } i = x+1, \dots, M_k - 1, \\
\tilde{\eta}_{M_k,k} &= 0, \qquad\qquad \text{if } M_k \neq x, \\
\tilde{\eta}_{M_k+1,k} &= \begin{cases} \tilde{\eta}_{M_k-1,k} + \tilde{\eta}_{M_k+1,k-1} & \text{if } \ M_k \geq x+2, \\ \frac{1}{2}\tilde{\eta}_{x+1,k-1} + \frac{1}{4}\tilde{\eta}_{x,k-1} + \tilde{\eta}_{x+2,k-1} & \text{if } \ M_k = x+1, \\ \frac{1}{2}\tilde{\eta}_{x,k-1} + \tilde{\eta}_{x+1,k-1} & \text{if } \ M_k = x. \end{cases}
\end{aligned} \tag{4.3.15}$$

We write for all $j \in \mathcal{C}(t+1)$, with $f_{ij}(k)$ implicitly defined by the coefficients in (4.3.15),

$$\tilde{\eta}_{j,k} = \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k)\tilde{\eta}_{i,k-1} + \mathbf{1}_{j \in \mathcal{C}'(t+1)}\tilde{\eta}_{j,k-1}. \tag{4.3.16}$$

Since we made an addition to a stable configuration, by Lemma 4.2.2, we only encounter configurations in $\tilde{\Omega}_N$. From a case by case analysis of (4.3.15), we claim that for all $j \in \mathcal{C}(t+1)$ we have

$$\mathcal{R}(\tilde{\eta}_{j,k}) = \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k)\mathcal{R}(\tilde{\eta}_{i,k-1}); \tag{4.3.17}$$

the reader can verify this for all cases. Note that for all $j \in \mathcal{C}(t+1)$, $\mathcal{R}(\tilde{\eta}_{j,k}) \neq a$. If $j \in \mathcal{C}'(t+1)$, then in the first wave j becomes full.

To prove the proposition, we start with the case $K = 1$, for which we have

$$\begin{aligned}
\eta_j(t+1) = \tilde{\eta}_{j,1} &= \sum_{i \in \mathcal{T}(t+1)} f_{ij}(1)\tilde{\eta}_{i,0} + \eta_j(t)\mathbf{1}_{j \in \mathcal{C}'(t+1)} \\
&= \sum_{i \in \mathcal{T}(t+1)} f_{ij}(1)\eta_i(t) + f_{xj}(1)U_{t+1} + \eta_j(t)\mathbf{1}_{j \in \mathcal{C}'(t+1)}.
\end{aligned}$$

We also have, according to (4.3.17),

$$\begin{aligned}
\mathcal{R}(\eta_j(t+1)) &= \sum_{i \in \mathcal{T}(t+1)} f_{ij}(1)\mathcal{R}(\tilde{\eta}_{i,0}) \\
&= f_{xj}(1) + \sum_{i \in \mathcal{T}(t+1)} f_{ij}(1).
\end{aligned}$$

Hence if $K = 1$ we choose $F_{ij} = f_{ij}(1)$ and part 1 of the proposition is proved for this case.

For $K > 1$, we use induction in k. Here we only consider sites that are not in $\mathcal{C}'(t+1)$; we already treated these above in the case $K = 1$. For wave $k-1$, we make the induction hypothesis that

$$\tilde{\eta}_{j,k-1} = \sum_{m \in \mathcal{T}(t+1)} F_{mj}(k-1)\eta_m(t) + F_{xj}(k-1)U_{t+1}, \tag{4.3.18}$$

with

$$\sum_{m \in \mathcal{T}(t+1)} F_{mj}(k-1) + F_{xj}(k-1) = \mathcal{R}(\tilde{\eta}_{j,k-1}). \tag{4.3.19}$$

Inserting this in (4.3.16), we get

$$\begin{aligned}
\tilde{\eta}_{j,k} &= \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k)\tilde{\eta}_{i,k-1} \\
&= \sum_{m \in \mathcal{T}(t+1)} \sum_{i \in \mathcal{T}(t+1)} F_{mi}(k-1)f_{ij}(k)\eta_m(t) + \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k)F_{xi}(k-1)U_{t+1},
\end{aligned}$$

and inserting this in (4.3.17), we get

$$\begin{aligned}
\mathcal{R}(\tilde{\eta}_{j,k}) &= \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k)\mathcal{R}(\tilde{\eta}_{i,k-1}) \\
&= \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k)\left[\sum_{m \in \mathcal{T}(t+1)} F_{mi}(k-1) + F_{xi}(k-1)\right].
\end{aligned}$$

Hence, if we define

$$F_{mj}(k) = \sum_{i \in \mathcal{T}(t+1)} f_{ij}(k) F_{mi}(k-1),$$

then (4.3.18) and (4.3.19) are also true for wave k. For $k-1=0$, the hypothesis is also true, with $F_{mi}(0) = \mathbf{1}_{m=i}$. We define $F_{ij} := F_{ij}(K)$, and then the first part of the proposition is proved for all K.

To prove part 2 of the proposition, we derive a lower bound for F_{xj}. The number K of waves in an avalanche is equal to the minimum of the distance to the end sites, leading to the upper bound $K \leq \lceil N/2 \rceil$.

After the first wave, (4.3.15) gives for all nonempty $j \neq x$, $F_{xj}(1) \geq (\frac{1}{2})^{N+1}$. At the start of the next wave, if there is one, the fraction of U_{t+1} present at x is equal to $F_{xx}(1) = \frac{1}{2}$. Hence, if after the second wave there is a third one, even if we ignore all fractions of U_t on sites other than x, then we still have, again by (4.3.15), $F_{xj}(2) > \frac{1}{2}(\frac{1}{2})^{N+1}$. So if before each wave we always ignore all fractions of U_t on sites other than x, and if we assume the maximum number of waves, then we arrive at a lower bound for nonempty sites j:

$$F_{xj} \geq (\frac{1}{2})^{\lceil N/2 \rceil - 1} (\frac{1}{2})^{N+1} \geq 2^{-\lceil 3N/2 \rceil}.$$

We now prove part 3. For $K=1$, part 3 of the proposition follows directly from (4.3.15), so we discuss the case $K > 1$. Moreover, we only discuss the case $j \geq x$, since by symmetry, the case $j \leq x$ is similar. We will show that for every $k \in \{1, \ldots, K-1\}$,

$$\frac{1}{2} F_{xx}(k) > F_{x,x+1}(k) > \cdots > F_{x,M_k-1}(k) = F_{x,M_k+1}(k), \tag{4.3.20}$$

and

$$F_{x,M_k+1}(k) \geq F_{x,M_{k-1}+1}(k-1), \tag{4.3.21}$$

where we define $F_{x,M_0+1}(0) = 0$. In the final wave, the sites $j > M_K + 1$ do not change, so the inequality in part 3 of the proposition for these sites follows from (4.3.21). Therefore, after proving (4.3.20) and (4.3.21) for all $k \in \{1, \ldots, K-1\}$, we will show that the required result follows for the sites $x \leq j \leq M_K$.

We will now prove (4.3.20) and (4.3.21) for all $k \in \{1, \ldots, K-1\}$, using induction in k. After the first wave, we have from (4.3.15) that

$$\frac{1}{2} F_{xx}(1) > F_{x,x+1}(1) > \cdots > F_{x,M_k-1}(1) = F_{x,M_k+1}(1),$$

so that (4.3.20) and (4.3.21) are satisfied after the first wave.

Now assume as induction hypothesis that (4.3.20) is true after wave k, with $k < K-1$. We have seen that this is true after the first wave. We rewrite (4.3.15), for every $k < K-1$, so that $M_{k+1} > x$. In the first line, we use that for all $k < K-1$, $F_{x,x+1}(k) = F_{x,x-1}(k)$. First, we explain why this is true.

By (4.3.15), we have in general, for positive and negative y, that $F_{x,x+y}(k)$ is a function of $F_{xz}(k-1)$, with $z \in \mathcal{Z}_1(y) = \{x+y-1, \ldots, x+y+1\}$, that is symmetric in y as long as all sites in $\mathcal{Z}_1(y) \cup \mathcal{Z}_1(-y)$ toppled in wave $k-1$. Continuing this reasoning, we have that $F_{x,x+y}(k)$ is a function of $F_{xz}(0)$, with $z \in \mathcal{Z}_k(y) = \{x+y-k, \ldots, x+y+k\}$, that is symmetric in y as long as all sites in $\mathcal{Z}_k(y) \cup \mathcal{Z}_k(-y)$ topple in the first wave. If $k < K-1$, this requirement is satisfied for all $x-1-k, \ldots, x+1+k$. Moreover, we have that $F_{xz}(0) = \mathbf{1}_{x=z}$, so that we obtain that $F_{x,x+1}(k) = F_{x,x-1}(k)$.

We now write, for every $k < K-1$,

$$
\begin{aligned}
F_{xx}(k+1) &= F_{x,x+1}(k) + \frac{1}{2}F_{xx}(k), \\
F_{x,x+1}(k+1) &= \begin{cases} \frac{1}{2}F_{x,x+2}(k) + \frac{1}{4}F_{xx}(k+1) & \text{if } M_{k+1} > x+1, \\ 0 & \text{if } M_{k+1} = x+1, \end{cases} \\
F_{x,x+i}(k+1) &= \frac{1}{2}F_{x,x+i+1}(k) + \frac{1}{2}F_{x,x+i-1}(k+1) \\
&\qquad\qquad\qquad \text{for } i = x+2, \ldots, M_{k+1}-1, \\
F_{xM_k}(k+1) &= 0, \\
F_{x,M_k+1}(k+1) &= \begin{cases} F_{x,M_k-1}(k+1) & \text{if } M_{k+1} > x+1, \\ \frac{1}{2}F_{x,M_k-1}(k+1) & \text{if } M_{k+1} = x+1. \end{cases}
\end{aligned}
\tag{4.3.22}
$$

If $M_{k+1} = x+1$, then (4.3.20) and (4.3.21) follow directly for $k+1$ from this expression. However, when $M_{k+1} > x+1$, we need the following derivation.

From (4.3.22) and the induction hypothesis, we find the following inequalities, each one following from the previous one:

$$F_{xx}(k+1) = F_{x,x+1}(k) + \frac{1}{2}F_{xx}(k) < \frac{1}{2}F_{xx}(k) + \frac{1}{2}F_{xx}(k) = F_{xx}(k),$$

$$F_{x,x+1}(k+1) = \frac{1}{2}F_{x,x+2}(k) + \frac{1}{4}F_{xx}(k+1) < \frac{1}{2}F_{x,x+1}(k) + \frac{1}{4}F_{xx}(k) = \frac{1}{2}F_{xx}(k+1),$$

If $M_k = x+2$, then $F_{x,x+2} = 0$, and (4.3.20) and (4.3.21) are satisfied. For $M_k > x+2$, we have

$$
\begin{aligned}
F_{x,x+2}(k+1) &= \frac{1}{2}F_{x,x+3}(k) + \frac{1}{2}F_{x,x+1}(k+1) \\
&< \frac{1}{2}F_{x,x+2}(k) + \frac{1}{4}F_{xx}(k+1) = F_{x,x+1}(k+1).
\end{aligned}
$$

For all $i = 2, \ldots, M_{k+1} - x - 1$, if $F_{x,x+i}(k+1) < F_{x,x+i-1}(k+1)$ then

$$\begin{aligned} F_{x,x+i+1}(k+1) &= \frac{1}{2}F_{x,x+i+2}(k) + \frac{1}{2}F_{x,x+i}(k+1) \\ &< \frac{1}{2}F_{x,x+i+1}(k) + \frac{1}{2}F_{x,x+i-1}(k+1) = F_{x,x+i}(k+1). \end{aligned} \tag{4.3.23}$$

Since $F_{x,x+i}(k+1) < F_{x,x+i-1}(k+1)$ is true for $i = 2$, (4.3.20) follows for wave $k+1$, and is thus proved for every $k < K$. Moreover, we have

$$F_{x,M_{k+1}+1}(k+1) = F_{x,M_{k+1}-1}(k+1) = \frac{1}{2}F_{x,M_{k+1}}(k) + \frac{1}{2}F_{x,M_{k+1}-2}(k+1).$$

With the above derived $F_{x,M_{k+1}-1}(k+1) < F_{x,M_{k+1}-2}(k+1)$, it follows that $F_{x,M_{k+1}}(k) < F_{x,M_{k+1}-2}(k+1)$, so that

$$F_{x,M_{k+1}-1}(k+1) > F_{x,M_{k+1}}(k) = F_{x,M_k+1}(k),$$

which is (4.3.21).

Finally we discuss the last wave. For the last wave, we need to discuss several cases. If $M_K = x$, then either $M'_k < x$ or $M'_K = x$, but if $M_K > x$, then $M'_K = x$, because at least one of the end sites of the last wave is x.

In case $M_K = M'_K = x$ we have $F_{xx}(K) = 0$, if $M_k = x$ and $M'_K < x$ then

$$F_{xx}(K) = \frac{1}{2}F_{x,x-1}(K-1) + \frac{1}{4}F_{xx}(K-1) = \frac{1}{2}F_{x,x+1}(K-1) + \frac{1}{4}F_{xx}(K-1),$$

In both cases, we have

$$F_{x,x+1}(K) = \frac{1}{2}F_{xx}(K-1) = F_{x,x+2}(K-1),$$

so that $F_{xx}(K) > F_{x,x+1}(K)$. Part 3 follows for $M_K = x$.

In case $M_K = x + 1$ we have

$$\begin{aligned} F_{xx}(K) &= \frac{1}{2}F_{x,x+1}(K-1) + \frac{1}{4}F_{xx}(K-1), \\ F_{x,x+1}(K) &= 0, \\ F_{x,x+2}(K) &= F_{xx}(K) = \frac{1}{2}F_{x,x+1}(K-1) + \frac{1}{4}F_{xx}(K-1) \\ &> \frac{1}{2}F_{x,x+1}(K-1) + \frac{1}{2}F_{x,x+1}(K-1) > F_{x,x+1}(K-1). \end{aligned}$$

For all $M_K > x+1$ we have

$$\begin{aligned} F_{xx}(K) &= \frac{1}{2}F_{x,x+1}(K-1) + \frac{1}{4}F_{xx}(K-1), \\ F_{x,x+1}(K) &= \frac{1}{2}F_{x,x+2}(K-1) + \frac{1}{4}F_{xx}(K-1) < F_{xx}(K). \end{aligned}$$

In this case, verifying (4.3.20) and (4.3.21) proceeds as in the previous derivation for the case $k < K, M_k \geq x+2$. □

4.3.3 Absolute continuity of one-site marginals of stationary distributions

Consider a one-site marginal ν_j of any stationary distribution ν of Zhang's sandpile model. It is easy to see that ν_j will have an atom at 0, because after each avalanche there remains at least one empty site. It is intuitively clear that there can be no other atoms: by only making uniformly distributed additions, it seems impossible to create further atoms. Here we prove the stronger statement that the one-site marginals of any stationary distribution are absolutely continuous with respect to Lebesgue measure on $(0,1)$.

Theorem 4.3.24. *Let ν be a stationary distribution for Zhang's model on N sites. Every one-site marginal of ν is on $(0,1)$ absolutely continuous with respect to Lebesgue measure.*

Proof. Let $A \subset (0,1)$ be so that $\lambda(A) = 0$, where λ denotes Lebesgue measure. We pick a starting configuration η according to ν. We define a stopping time τ as the first time t such that all non-zero energies $\eta_i(t)$ contain a nonzero contribution of at least one of the added amounts $U_1, U_2, \ldots, U_t$. We then write, for an arbitrary nonzero site j,

$$\mathbb{P}_\nu(\eta_j(t) \in A) \leq \mathbb{P}_\nu(\eta_j(t) \in A, \tau < t) + \mathbb{P}_\nu(t \leq \tau). \tag{4.3.25}$$

The second term at the right hand side tends to 0 as $t \to \infty$ because by Remark 4.3.7, a.s. within finite time each site has participated in an avalanche at least once, and by Proposition 4.3.12 part 2, each site contains a nonzero contribution of the addition that started the last avalanche it participated in.

We claim that the first term at the right hand side is equal to zero. To this end, we first observe that $\eta_j(t)$ is built up of fractions of $\eta_i(0)$, $i = 1, \ldots, N$, and the additions $U_1, U_2, \ldots, U_t$. These fractions are random variables themselves, and we can bound this term by

$$\mathbb{P}_\nu\left(\sum_{i=1}^{N} Z_i\eta_i(0) + \sum_{s=1}^{t} Y_sU_s \in A, \sum_{s=1}^{t} Y_s > 0\right), \tag{4.3.26}$$

where Z_i represents the (random) fraction of $\eta_i(0)$ in $\eta_j(t)$, and Y_s represents the (random) fraction of U_s in $\eta_j(t)$.

We clearly have that the U_s are all independent of each other and of $\eta_i(0)$ for all i. However, the U_s are not necessarily independent of the Z_i and the Y_s, since the numerical value of the U_s affects the relevant fractions. Also, we know from the analysis in the previous subsection that the Z_i and Y_s can only take non-negative values in a countable set. Summing over all elements in this set, we rewrite (4.3.26) as

$$\sum_{z_i, y_s; \sum_s y_s > 0} \mathbb{P}_\nu \left(\sum_{i=1}^N z_i \eta_i(0) + \sum_{s=1}^t y_s U_s \in A, Z_i = z_i, Y_s = y_s \right)$$

which is at most

$$\sum_{z_i, y_s; \sum_s y_s > 0} \mathbb{P}_\nu \left(\sum_{i=1}^N z_i \eta_i(0) + \sum_{s=1}^t y_s U_s \in A \right),$$

which, by the independence of the U_s and the $\eta_i(0)$, is equal to

$$\sum_{z_i, y_s; \sum_s y_s > 0} \int \mathbb{P}_\nu \left(\sum_{i=1}^N z_i x_i + \sum_{s=1}^t y_s U_s \in A \right) d\nu(x_1, \dots, x_N).$$

Since $\sum_{s=1}^t y_s > 0$, U_s are independent uniforms, and by assumption $\lambda(A) = 0$, the probabilities inside the integral are clearly zero. Since the left hand side of (4.3.25) is equal to $\nu_j(A)$ for all t, we now take the limit $t \to \infty$ on both sides, and we conclude that $\nu_j(A) = 0$. $\square$

Remark 4.3.27. The same proof shows that for every stationary measure ν, and for every $i_1, \dots, i_k \in \{1, \dots, N\}$, conditional on all sites $i_1, \dots, i_k$ being nonempty, the joint distribution of $\eta_{i_1}, \dots, \eta_{i_k}$ under ν is absolutely continuous with respect to Lebesgue measure on $(0,1)^k$.

4.4 The $(1,[a,b])$-model

In this section we consider the simplest version of Zhang's model: the $(1,[a,b])$-model. In words: there is only one site and we add amounts of energy that are uniformly distributed on the interval $[a,b]$, with $0 \le a < b \le 1$.

4.4.1 Uniqueness of the stationary distribution

Before turning to the particular case $a = 0$, we prove uniqueness of the stationary distribution for all $[a,b] \subseteq [0,1]$. We also prove that every initial distribution on Ω_1

converges to this stationary measure. We find two different kinds of convergence; convergence in total variation is the strongest, but we cannot obtain this for all values of a and b.

Theorem 4.4.1. *(a) The $(1, [a, b])$ model has a unique stationary distribution $\rho = \rho^{ab}$. For every initial distribution $\mathbb{P}_\eta$ on Ω_1, we have time-average total variation convergence to ρ, i.e.,*

$$\lim_{t\to\infty} \sup_{A\subset\Omega_1} \left| \frac{1}{t} \sum_{s=0}^{t} \mathbb{P}_\eta \left(\eta(s) \in A\right) - \rho(A) \right| = 0.$$

(b) In addition, if there exists no integer $m > 1$ such that $[a, b] \subseteq [\frac{1}{m}, \frac{1}{m-1}]$, (hence in particular if $a = 0$), then we have convergence in total variation to ρ for every initial distribution $\mathbb{P}_\eta$ on Ω_1, i.e.,

$$\lim_{t\to\infty} \sup_{A\subset\Omega_1} |\mathbb{P}_\eta(\eta(t) \in A) - \rho(A)| = 0.$$

Proof. We prove this theorem by constructing a coupling. The two processes to be coupled have initial configurations η^1 and η^2, with η^1,$\eta^2 \in \Omega_1$. We denote by $\eta^1(t)$, $\eta^2(t)$ two independent copies of the process starting from η^1 and η^2 respectively. The corresponding independent additions at each time step are denoted by U_t^1 and U_t^2, respectively. Let $T_1 = \min\{t : \eta^1(t) = 0\}$ and $T_2 = \min\{t : \eta^2(t) = 0\}$. Suppose (without loss of generality) that $T_2 \geq T_1$. We define a shift-coupling ([48], Chapter 5) as follows:

$$\begin{aligned} \hat{\eta}^1(t) &= \eta^1(t) && \text{for all } t, \\ \hat{\eta}^2(t) &= \begin{cases} \eta^2(t) & \text{for } t < T_2, \\ \eta^1(t - (T_2 - T_1)) & \text{for } t \geq T_2. \end{cases} \end{aligned}$$

Defining $T = \min\{t : \eta^1(t) = \eta^2(t) = 0\}$, we also define the exact coupling

$$\begin{aligned} \hat{\eta}^1(t) &= \eta^1(t) && \text{for all } t, \\ \hat{\eta}^3(t) &= \begin{cases} \eta^2(t) & \text{for } t < T, \\ \eta^1(t) & \text{for } t \geq T. \end{cases} \end{aligned}$$

Since the process is Markov, both couplings have the correct distribution. We write $\tilde{\mathbb{P}} = \mathbb{P}_{\eta_1} \times \mathbb{P}_{\eta_2}$. Since $\tilde{\mathbb{P}}(T_2 < \infty) = \tilde{\mathbb{P}}(T_1 < \infty) = 1$, the shift-coupling is always successful, and (a) follows.

To investigate whether $\eta^1(t) = \eta^2(t) = 0$ occurs infinitely often $\tilde{\mathbb{P}}$-a.s., we define $\mathcal{N} = \{n : (n-1)a < 1, nb > 1\}$; this is the set of possible numbers of time steps between successive events $\eta^1(t) = 0$. In words, an $n \in \mathcal{N}$ is such that, starting from $\eta^1 = 0$, it is possible that in $n-1$ steps we do not yet reach energy 1, but in n steps we do. To give an example, if $a \geq 1/2$, then $\mathcal{N} = \{2\}$.

If the gcd of $\mathcal{N}$ is 1 (this is in particular the case if $a = 0$), then the processes $\{t : \eta^1(t) = 0\}$ and $\{t : \eta^2(t) = 0\}$ are independent aperiodic renewal processes, and it follows that $\eta^1(t) = \eta^2(t) = 0$ happens infinitely often $\tilde{\mathbb{P}}$-a.s.

As we have seen, for $a > 0$, the gcd of $\mathcal{N}$ need not be 1. In fact, we can see from the definition of $\mathcal{N}$ that this is the case if (and only if) there is an integer $m > 1$ such that $[a, b] \subseteq [\frac{1}{m}, \frac{1}{m-1}]$. Then $\mathcal{N} = \{m\}$. For such values of a and b, the processes $\{t : \eta^1(t) = 0\}$ and $\{t : \eta^2(t) = 0\}$ are periodic, so that we do not have a successful exact coupling. □

4.4.2 The stationary distribution of the $(1,[0,b])$-model

We write ρ^b for the stationary measure ρ^{0b} of the $(1,[0,b])$-model and F^b for the distribution function of the amount of energy at stationarity, that is,

$$F^b(h) = \rho^b(\eta : 0 \leq \eta \leq h).$$

We prove the following explicit solution for $F^b(h)$.

Theorem 4.4.2. *(a) The distribution function of the energy in the $(1,[0,b])$-model at stationarity is given by*

$$F^b(h) = \begin{cases} 0 & \text{for } h < 0, \\ F^b(0) > 0 & \text{for } h = 0, \\ F^b(0)\sum_{\kappa=0}^{m_h} \frac{(-1)^\kappa}{b^\kappa \kappa!}(h-\kappa b)^\kappa e^{\frac{h-\kappa b}{b}} & \text{for } 0 < h \leq 1, \\ 1 & \text{for } h > 1, \end{cases} \tag{4.4.3}$$

where $m_h = \lceil \frac{h}{b} \rceil - 1$ and where

$$F^b(0) = \frac{1}{\sum_{\kappa=0}^{m_1} \frac{(-1)^\kappa}{b^\kappa \kappa!}(1-\kappa b)^\kappa e^{\frac{1-\kappa b}{b}}}$$

follows from the identity $F^b(1) = 1$.

(b) For $h \in [0,1]$ we have

$$\lim_{b \to 0} F^b(h) = h.$$

We remark that although in (a) we have a more or less explicit expression for $F^b(h)$, the convergence in (b) is not proved analytically, but rather probabilistically.

Proof of Theorem 4.4.2, part (a). Observe that the process for one site is defined as

$$\eta(t+1) = (\eta(t) + U_{t+1}) \ \mathbf{1}_{\eta(t)+U_{t+1}<1}. \tag{4.4.4}$$

We define $F_t^b(h) = \mathbb{P}(\eta(t) \leq h)$, and derive an expression for $F_{t+1}^b(h)$ in terms of $F_t^b(h)$. In the stationary situation, these two functions should be equal. We deduce from (4.4.4) that for $0 \leq h \leq 1$,

$$F_{t+1}^b(h) = \mathbb{P}(\eta(t) + U_{t+1} < h) + \mathbb{P}(\eta(t) + U_{t+1} \geq 1). \tag{4.4.5}$$

We compute for $0 \leq h \leq b$,

$$\begin{aligned}
\mathbb{P}(\eta(t) + U_{t+1} \leq h) &= \mathbb{P}(\eta(t) \leq h - U_{t+1}) \\
&= \int_0^h \frac{1}{b}\mathbb{P}(\eta(t) \leq h-u)\, du \\
&= \int_0^h \frac{1}{b} F_t^b(h-u)\, du,
\end{aligned} \tag{4.4.6}$$

and likewise for $b \leq h \leq 1$ we find

$$\mathbb{P}(\eta(t) + U_{t+1} \leq h) = \int_0^b \frac{1}{b}(F_t^b(h-u))\, du. \tag{4.4.7}$$

Finally, using that $F^b(1) = 1$ (this follows from (4.4.4)),

$$\begin{aligned}
\mathbb{P}(\eta(t) + U_{t+1} \geq 1) &= \int_0^b \frac{1}{b}(F_t^b(1) - F_t^b(1-u))\, du \\
&= \int_0^b \frac{(1 - F_t^b(1-u))}{b}\, du = F_{t+1}^b(0).
\end{aligned} \tag{4.4.8}$$

Putting (4.4.5), (4.4.6), (4.4.7) and (4.4.8) together leads to the conclusion that the stationary distribution $F^b(h)$ satisfies

$$F^b(h) = \begin{cases} \int_0^h \frac{F^b(h-u)}{b}\, du + F^b(0) & \text{if } 0 \leq h \leq b, \\ \int_0^b \frac{F^b(h-u)}{b}\, du + F^b(0) & \text{if } b \leq h \leq 1. \end{cases} \tag{4.4.9}$$

Furthermore, since $F^b(h)$ is a distribution function, $F^b(h) = 0$ for $h < 0$ and $F^b(h) = 1$ for $h > 1$. We can rewrite equation (4.4.9) as a differential delay equation. Let $f^b(h)$ be a density corresponding to F^b for $0 < h < 1$; this density exists according to Theorem 4.3.24.

We differentiate (4.4.9) twice on both sides, to get in the case $0 < h < b$,

$$\frac{df^b(h)}{dh} = \frac{1}{b} f^b(h),$$

and in the case $b < h \le 1$,

$$f^b(h) = \frac{1}{b}(F^b(h) - F^b(h-b)). \tag{4.4.10}$$

At this point, we can conclude that the solution is unique and could in principle be found using the method of steps. However, since we already have the candidate solution given in Theorem 4.4.2, we only need to check that it indeed satisfies equation (4.4.9).

In the case $0 < h < b$, in which case $m_h = 0$, we have $F^b(h) = F^b(0)e^{\frac{h}{b}}$, which is consistent with Theorem 4.4.2.

We check that for the derivative f^b of F^b as defined in (4.4.3), for $b \le h \le 1$,

$$\begin{aligned} f^b(h) &= -\frac{F^b(0)}{b}\sum_{\kappa=1}^{m_h}(-\frac{1}{b})^{\kappa-1}\frac{1}{(\kappa-1)!}(h-\kappa b)^{\kappa-1}e^{\frac{h-\kappa b}{b}} \\ &\quad +\frac{F^b(0)}{b}\sum_{\kappa=0}^{m_h}(-\frac{1}{b})^{\kappa}\frac{1}{\kappa!}(h-\kappa b)^{\kappa}e^{\frac{h-\kappa b}{b}}, \end{aligned}$$

whereas

$$\frac{F^b(h)}{b} = \frac{F^b(0)}{b}\sum_{\kappa=0}^{m_h}(-\frac{1}{b})^{\kappa}\frac{1}{\kappa!}(h-\kappa b)^{\kappa}e^{\frac{h-\kappa b}{b}}$$

and

$$\begin{aligned} -\frac{F^b(h-b)}{b} &= -\frac{F^b(0)}{b}\sum_{\kappa=0}^{m_h-1}(-\frac{1}{b})^{\kappa}\frac{1}{\kappa!}\kappa(h-(\kappa+1)b)^{\kappa}e^{\frac{h-(\kappa+1)b}{b}} \\ &= -\frac{F^b(0)}{b}\sum_{\kappa=1}^{m_h}(-\frac{1}{b})^{\kappa-1}\frac{1}{(\kappa-1)!}(h-\kappa b)^{\kappa-1}e^{\frac{h-\kappa b}{b}}, \end{aligned}$$

which leads to (4.4.10) as required. □

We remark that the probability density function $f^b(h)$ has an essential point of discontinuity at $h = b$. Figures 4.1 and 4.2 show two examples of $f^b(h)$.

Proof of Theorem 4.4.2, part (b). For a while, we fix $b > 0$ and write $\eta^b(t)$ for the state of our process when we start with $\eta^b(0) = 0$. For fixed h, define a process $X^b(t)$ by $X^b(t) = 1$ if $\eta^b(t) \le h$ and $X^b(t) = 0$ if $\eta^b(t) > h$. The process X^b is a delayed renewal process; a renewal occurs at t if $X^b(t-1) = 1$ and $X^b(t) = 0$. Let $K(t)$ be the number of renewals up to (and including) time t, where we take $K(0) = 0$. The k-th renewal takes place at time T_k, where we define $T_0 = 0$. The number of indices $t \in [T_{k-1}, T_k)$ with $X^b(t) = 0$ is denoted by $Z^b(k)$; the number

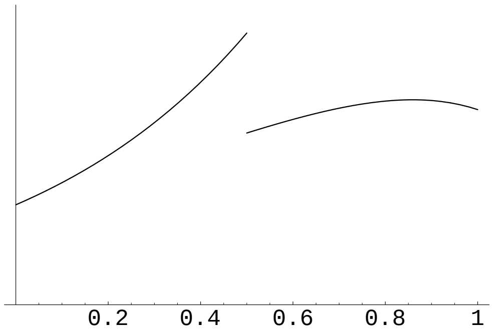

Figure 4.1: $f^b(h)$ for $b = \frac{1}{2}$. Note the discontinuity at $h = \frac{1}{2}$.

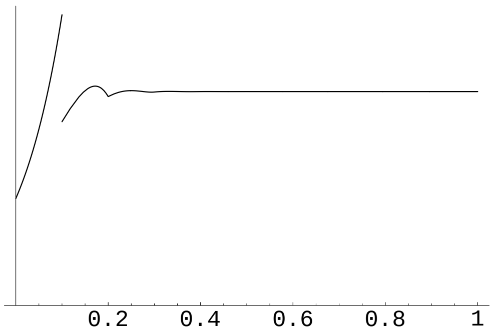

Figure 4.2: $f^b(h)$ for $b = \frac{1}{10}$. This figure illustrates that for small b, $f^b(h)$ tends to the uniform distribution.

of indices t in that interval with $X^b(t) = 1$ by $W^b(k)$. Typical random variables with these distributions are denoted by Z^b and W^b respectively.

By the identity

$$\sum_{k=1}^{K(t-1)} W^b(k) \leq \sum_{i=0}^{t-1} X^b(i) \leq \sum_{j=1}^{K(t-1)} W^b(j) + W^b(K(t-1)+1)$$

and the well known fact that $K(t)/t \to 1/E(Z^b + W^b)$ as $t \to \infty$, we find that

$$\begin{aligned} F^b(h) &= \lim_{t\to\infty} \mathbb{P}(\eta^b(t) \leq h) = \lim_{t\to\infty} \mathbb{P}(X^b(t) = 1) \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} X^b(i) \\ &= \lim_{t\to\infty} \frac{K(t)}{t} E(W^b) \\ &= \frac{E(W^b)}{E(Z^b + W^b)}. \end{aligned}$$

To compute these expectations in the limit $b \to 0$, we use another renewal process. Let $U_1, U_2, \dots$ be independent uniform random variables on $[0,1]$ and write $S_n^1 = U_1 + \cdots + U_n$. We define

$$N(s) = \max\{n \in \mathbb{N} : S_n^1 \leq s\}.$$

The process $\{N(s) : s \geq 0\}$ is a renewal process, and

$$\lim_{s\to\infty} \frac{\mathbb{E}(N(s))}{s} = \mathbb{E}(U_1)^{-1} = 2. \tag{4.4.11}$$

Now observe that W^b and $N(h/b)-1$ have the same distribution, and that Z^b+W^b has the same distribution as $N(1/b) - 1$. Hence

$$\lim_{b\to 0} \frac{E(W^b)}{E(Z^b + W^b)} = \lim_{b\to 0} \frac{E(N(h/b) - 1)}{E(N(1/b) - 1)},$$

and this last limit is equal to h, according to (4.4.11). □

4.5 The $(N,[a,b])$-model with $N \geq 2$ and $a \geq \frac{1}{2}$

4.5.1 Uniqueness of stationary distribution

In the course of the process of Zhang's model, the energies of all sites can be randomly augmented through additions, and randomly redistributed among other sites through avalanches. Thus at time t, every site contains some linear combination of all additions up to time t, and the energies at time 0. In a series of lemma's we derive very detailed properties of these combinations in the case $a \geq 1/2$. These properties are crucial to prove the following result.

Theorem 4.5.1. *The $(N,[a,b])$ model with $a \geq \frac{1}{2}$, has a unique stationary distribution $\mu = \mu^{ab}$. For every initial distribution ν on Ω_N, $\mathbb{P}_\nu$ converges exponentially fast in total variation to μ.*

We have demonstrated for the case $a \geq \frac{1}{2}$ that after a finite (random) time, we only encounter regular configurations (Proposition 4.3.9). By Lemma 4.3.10, if $\eta(t-1)$ is regular, then the knowledge of $\mathcal{R}(\eta(t-1))$ and X_t suffices to know the number of topplings of each site at time t. From these numbers, we can infer the endsites of all waves, and then by repeatedly applying (4.3.15), we can find the factors F_{ij} in Proposition 4.3.12. Thus, also these factors are functions of $\mathcal{R}(\eta(t-1))$ and X_t only. Using this observation, we prove the following.

Lemma 4.5.2. *Let $a \geq \frac{1}{2}$. Suppose at some (possibly random) time τ we have a configuration $\xi(\tau)$ with no anomalous sites. Then for all $j = 1, \dots, N$ and for $t \geq \tau$, we can write*

$$\xi_j(t) = \sum_{\theta=\tau+1}^{t} A_{\theta j}(t) U_\theta + \sum_{m=1}^{N} B_{mj}(t) \xi_m(\tau) \tag{4.5.3}$$

in such a way that the coefficients in (4.5.3) *satisfy*

$$A_{\theta j}(t) = A_{\theta j}(\mathcal{R}(\xi(\tau)), X_{\tau+1}, \dots, X_t)$$

and

$$B_{mj}(t) = B_{mj}(\mathcal{R}(\xi(\tau)), X_{\tau+1}, \dots, X_t),$$

and such that for every j and every $t \geq \tau$,

$$\sum_{\theta=\tau+1}^{t} A_{\theta j}(t) + \sum_{m=1}^{N} B_{mj}(t) = \mathcal{R}(\xi_j(t)) = \mathbf{1}_{\xi_j(t) \neq 0}.$$

Remark 4.5.4. Notice that, in the special case that τ is a stopping time, $A_{\theta j}(t)$ is independent of the amounts added after time τ, i.e., $A_{\theta j}(t)$ and $\{U_\theta, \theta \geq \tau + 1\}$ are independent. We will make use of this observation in Section 4.5.2.

Proof of Lemma 4.5.2. We use induction. We start at $t = \tau$, where we choose $B_{mj}(\tau) = \mathcal{R}(\xi(\tau)_j)\, \mathbf{1}_{m=j}$, and $A_{\tau j}(t) = 0$ for all $t \geq \tau$ and for all j. We then have $\sum_{m=1}^{N} B_{mj}(\tau) = \mathcal{R}(\xi(\tau)_j)$, so that at $t = \tau$ the statement in the lemma is true. We next show that if the statement in the lemma is true at time $t \geq \tau$, then it is also true at time $t + 1$.

At time t we have for every $j = 1, \dots, N$,

$$\xi_j(t) = \sum_{\theta=\tau+1}^{t} A_{\theta j}(t) U_\theta + \sum_{m=1}^{N} B_{mj}(t) \xi_m(\tau),$$

with

$$\sum_{\theta=\tau+1}^{t} A_{\theta j}(t) + \sum_{m=1}^{N} B_{mj}(t) = \mathcal{R}(\xi(t)_j),$$

where all $A_{\theta j}(t)$ and $B_{mj}(t)$ are determined by $\mathcal{R}(\xi(\tau)), X_{\tau+1}, \ldots, X_t$, so that $\mathcal{R}(\xi(t))$ is also determined by $\mathcal{R}(\xi(\tau)), X_{\tau+1}, \ldots, X_t$. We first discuss the case where we added to a full site, so that an avalanche is started. In that case, the knowledge of $\mathcal{R}(\xi(\tau)), X_{\tau+1}, \ldots, X_{t+1}$ determines the sets $\mathcal{C}(t+1)$, $\mathcal{T}(t+1)$ and the factors F_{ij} from Proposition 4.3.12 (since $a \geq \frac{1}{2}$). We write, denoting $X_{t+1} = x$,

$$\begin{aligned}
\xi_j(t+1) &= \sum_{i \in \mathcal{T}(t+1)} F_{ij}\xi_i(t) + F_{xj}U_{t+1} \\
&= \sum_{i \in \mathcal{T}(t+1)} F_{ij}\left[\sum_{\theta=1}^{t} A_{\theta i}(t)U_\theta + \sum_{m=1}^{N} B_{mi}(t)\xi_m(\tau)\right] + F_{xj}U_{t+1}.
\end{aligned} \tag{4.5.5}$$

Thus we can identify

$$A_{\theta j}(t+1) = \sum_{i \in \mathcal{T}(t+1)} F_{ij}A_{\theta i}(t), \tag{4.5.6}$$

$$B_{mj}(t+1) = \sum_{i \in \mathcal{T}(t+1)} F_{ij}B_{mi}(t),$$

and

$$A_{t+1,j}(t+1) = F_{xj},$$

so that indeed all $A_{\theta j}(t+1)$ and $B_{mj}(t+1)$ are functions of $\mathcal{R}(\xi(\tau)), X_{\tau+1}, \ldots, X_{t+1}$ only. Furthermore,

$$\begin{aligned}
\sum_{\theta=1}^{t+1} A_{\theta j}(t+1) + \sum_{m=1}^{N} B_{mj}(t+1) &= \sum_{i \in \mathcal{T}(t+1)} F_{ij}\left[\sum_{\theta=1}^{t} A_{\theta i}(t) + \sum_{m=1}^{N} B_{mi}(t)\right] + F_{xj} \\
&= \sum_{i \in \mathcal{T}(t+1)} F_{ij} + F_{xj} = \mathcal{R}(\eta(t+1)_j),
\end{aligned} \tag{4.5.7}$$

where we used that by Lemma 4.3.10 all sites that toppled must have been full, therefore had reduced value 1.

If no avalanche was started, then the only site that changed is the addition site x, and it must have been empty at time t. Therefore, we have for all $\tau < \theta < t+1$,

$A_{\theta x}(t+1) = A_{\theta x}(t) = 0$, for all m, $B_{mx}(t+1) = B_{mx}(t) = 0$ and $A_{t+1,x}(t+1) = 1$, so that the above conclusion is the same. $\square$

For every θ, we have $\sum_{i \in \mathcal{T}(t+1)} A_{\theta i}(t) \leq 1$, because the addition U_θ gets redistributed by avalanches, but some part disappears through topplings of boundary sites. One might expect that, as an addition gets redistributed multiple times and many times some parts disappear at the boundary, the entire addition eventually disappears, and similarly for the energies $\xi_j(\tau)$. Indeed, we have the following results about the behavior of $A_{\theta i}(t)$ for fixed θ, and about the behavior of $B_{mj}(t)$ for fixed m.

Lemma 4.5.8. *For every θ, and for $t > \theta$,*

1. $\max_{1 \leq i \leq N} A_{\theta i}(t)$ *and* $\max_{1 \leq i \leq N} B_{mi}(t)$ *are both non-increasing in t.*
2. *For all θ, m and i,* $\lim_{t \to \infty} A_{\theta i}(t) = 0$, *and* $\lim_{t \to \infty} B_{mi}(t) = 0$.

Proof. We can assume that $t > \theta$. The proofs for $A_{\theta j}(t)$ and for $B_{mj}(t)$ proceed along the same line, so we will only discuss $A_{\theta j}(t)$. We will show that for every j, $A_{\theta j}(t+1) \leq \max_i A_{\theta i}(t)$, by considering one fixed j. If the energy of site j did not change in an avalanche at time $t+1$, then

$$A_{\theta j}(t+1) = A_{\theta j}(t) \leq \max_i A_{\theta i}(t).$$

If site j became empty in the avalanche, then

$$A_{\theta j}(t+1) = 0 < \max_i A_{\theta i}(t).$$

For the third possibility - the energy of site j changed to a nonzero value in an avalanche at time $t+1$ - we use (4.5.6), and estimate

$$A_{\theta j}(t+1) = \sum_{i \in \mathcal{T}(t+1)} F_{ij} A_{\theta i}(t) \leq \max_i A_{\theta i}(t) \sum_{i \in \mathcal{T}(t+1)} F_{ij}.$$

By Proposition 4.3.12 part (1) and (2), $\sum_{i \in \mathcal{T}(t+1)} F_{ij} \leq 1 - 2^{-\lceil 3N/2 \rceil}$, so that in this third case,

$$A_{\theta j}(t+1) \leq (1 - 2^{-\lceil 3N/2 \rceil}) \max_i A_{\theta i}(t) < \max_i A_{\theta i}(t).$$

Thus, it follows that $\max_i A_{\theta i}(t+1)$ can never be larger than $\max_i A_{\theta i}(t)$. This proves part (1). It also follows that when between t and $t+t'$ all sites have changed at least once, we are sure that $\max_i A_{\theta i}(t+t') \leq (1 - 2^{-\lceil 3N/2 \rceil}) \max_i A_{\theta i}(t)$.

We next derive an upper bound for the time that one of the sites can remain unchanged. Suppose at some finite time t when $\eta(t)$ is regular (see Proposition

4.3.9), we try never to change some sites again. If all sites are full, then this is impossible: the next avalanche will change all sites. If there is an empty site x, then the next addition (if it is not at site x) changes either the sites $1, \ldots, x$, or the sites $x, \ldots, N$. In the first case, after the avalanche we have a new empty site $x' < x$. If we keep trying not to change the sites $x, \ldots, N$, we have to keep making additions that move the empty site closer to the boundary (site 1). It will therefore reach the boundary in at most $N-1$ time steps. Then we have no choice but to change all sites: we can either add to the empty site and obtain the full configuration, so that with the next addition all sites will change, or add to any other site, which immediately changes all sites. This argument shows that the largest possible number of time steps between changing all sites is $N+1$. We therefore have

$$\max_i A_{\theta i}(t) < (1 - 2^{-\lceil 3N/2 \rceil})^{\lfloor \frac{t-\theta}{N+1} \rfloor}, \tag{4.5.9}$$

so that

$$\lim_{t \to \infty} \max_i A_{\theta i}(t) < \lim_{t \to \infty} (1 - 2^{-\lceil 3N/2 \rceil})^{\lfloor \frac{t-\theta}{N+1} \rfloor} = 0.$$

□

With the above results, we can now prove uniqueness of the stationary distribution.

Proof of Theorem 4.5.1. By compactness, there is at least one stationary measure μ. To prove the theorem, we will show that there is a coupling $(\hat{\eta}^1(t), \hat{\eta}^2(t))_0^\infty$ with probability law $\hat{\mathbb{P}}_{(\eta^1,\eta^2)}$ for two realizations of the $(N,[a,b])$ model with $a \geq \frac{1}{2}$, such that for all $\epsilon > 0$, and for all starting configurations η^1 and η^2, for $t \to \infty$ we have

$$\hat{\mathbb{P}}_{(\eta^1,\eta^2)} \left(\max_j |\hat{\eta}^1_j(t) - \hat{\eta}^2_j(t)| > \epsilon \right) = O(e^{-\alpha_N t})), \tag{4.5.10}$$

with $\alpha_N > 0$.

From (4.5.10), it follows that the Wasserstein distance ([16], Chapter 11.8) between any two measures μ_1 and μ_2 on Ω_N vanishes exponentially fast as $t \to \infty$. If we choose η^1 distributed according to μ stationary, then it is clear that every other measure on Ω_N converges exponentially fast to μ. In particular, it follows that μ is unique.

As in the proof of Theorem 4.4.1, the two processes to be coupled have initial configurations η^1 and η^2, with η^1,$\eta^2 \in \Omega_N$. The independent additions at each time step are denoted by U^1_t and U^2_t, the addition sites X^1_t and X^2_t.

We define the coupling as follows:

$$\begin{aligned} \hat{\eta}^1(t) &= \eta^1(t) && \text{for all } t \\ \hat{\eta}^2(t) &= \begin{cases} \eta^2(t) & \text{for } t \leq T, \\ \mathcal{A}_{U^1_t, X^1_t}(\hat{\eta}^2(t-1)) & \text{for } t > T, \end{cases} \end{aligned}$$

where $T = \min\{t > T' : \mathcal{R}(\eta^1(t)) = \mathcal{R}(\eta^2(t))\}$, and T' the first time that both $\eta^1(t)$ and $\eta^2(t)$ are regular. In Proposition 4.3.9 it was proven that $T' \leq N(N-1)$, uniformly in η. In words, this coupling is such that from the first time on where the reductions of $\hat{\eta}^1(t)$ and $\hat{\eta}^2(t)$ are the same, we make additions to both copies in the same manner, i.e., we add the same amounts at the same location to both copies. Then, by Lemma 4.3.10, in both copies the same avalanches will occur. We will then use Lemma 4.5.8 to show that, from time T on, the difference between $\hat{\eta}^1(t)$ and $\hat{\eta}^2(t)$ vanishes exponentially fast.

First we show that $\hat{\mathbb{P}}_{(\eta^1,\eta^2)}(T > t)$ is exponentially decreasing in t. There are $N+1$ possible reduced regular configurations. Once $\hat{\eta}^1(t)$ is regular, the addition site X^1_{t+1} uniquely determines the new reduced regular configuration $\mathcal{R}(\eta^1(t+1))$. This new reduced configuration cannot be the same as $\mathcal{R}(\eta^1(t))$. Thus, there are N equally likely possibilities for $\mathcal{R}(\eta^1(t+1))$, and likewise for $\mathcal{R}(\eta^2(t+1))$.

If $\mathcal{R}(\eta^1(t)) \neq \mathcal{R}(\eta^2(t))$, then one of the possibilities for $\mathcal{R}(\eta^1(t+1))$ is the same as $\mathcal{R}(\eta^2(t))$, so that there are $N-1$ possible reduced configurations that can be reached both from $\eta^1(t)$ and $\eta^2(t)$. The probability that $\mathcal{R}(\eta^1(t+1))$ is one of these is $\frac{N-1}{N}$, and the probability that $\mathcal{R}(\eta^2(t+1))$ is the same is $\frac{1}{N}$. Therefore, T is geometrically distributed, with parameter $p_N = \frac{N-1}{N^2}$.

We now use Lemma 4.5.2 with $\tau = T$. For $t > T$, we have in this case that $A^1_{j\theta}(t) = A^2_{j\theta}(t)$ and $B^1_{jm}(t) = B^2_{jm}(t)$, because from time T on, in both processes the same avalanches occur. Also, for $t > T$, we have chosen $U^1(t) = U^2(t)$. Therefore, for $t > T$,

$$\hat{\eta}^1_j(t) - \hat{\eta}^2_j(t) = \sum_{m=1}^{N} B^1_{jm}(t)\left(\hat{\eta}^1_m(T) - \hat{\eta}^2_m(T)\right).$$

From (4.5.9) in the proof of Lemma 4.5.8 we know that

$$B^1_{jm}(t) \leq (1 - 2^{-\lceil 3N/2 \rceil})^{\lfloor \frac{t-T}{N+1} \rfloor},$$

so that

$$\sum_{m=1}^{N} B^1_{jm}(t)\hat{\eta}^1_m(T) \leq N(1 - 2^{-\lceil 3N/2 \rceil})^{\lfloor \frac{t-T}{N+1} \rfloor},$$

so that for $t > T$, we arrive at

$$\max_j |\hat{\eta}^1_j(t) - \hat{\eta}^2_j(t)| \leq 2N(1 - 2^{-\lceil 3N/2 \rceil})^{\frac{t-T-1}{N+1}}.$$

We now split $\hat{\mathbb{P}}_{(\eta^1,\eta^2)}(\max_j |\hat{\eta}^1_j(t) - \hat{\eta}^2_j(t)| > \epsilon)$ into two terms, by conditioning on $t < 2T$ and $t \geq 2T$ respectively. Both terms decrease exponentially in t: the first term because the probability of $t < 2T$ is exponentially decreasing in t, and the second term because for $t \geq 2T$, $\max_j |\hat{\eta}_j^1(t) - \hat{\eta}_j^2(t)|$ itself is exponentially decreasing in t. □

A comparison of the two terms $\mathbb{P}(t < 2T)$ and $\max_j |\hat{\eta}_j^1(t) - \hat{\eta}_j^2(t)|$ yields that for N large, the second term dominates. We find that α_N depends, for large N, on N as $\alpha_N = -\frac{1}{2}\ln(1 - 2^{-\lceil 3N/2 \rceil})^{\frac{1}{N+1}}$. We see that as N increases, our bound on the speed of convergence decreases exponentially fast to zero.

4.5.2 Emergence of quasi-units in the infinite volume limit

In Proposition 4.3.4, we already noticed a close similarity between the stationary distribution of Zhang's model with $a \geq 1/2$, and the abelian sandpile model. We found that the stationary distribution of the reduced Zhang's model, in which we label full sites as 1 and empty sites as 0, is equal to that of the abelian sandpile model (Proposition 4.3.9).

In this section, we find that in the limit $N \to \infty$, the similarity is even stronger. We find emergence of Zhang's quasi-units in the following sense: as $N \to \infty$, all one-site marginals of the stationary distribution concentrate on a single, nonrandom value. We believe that the same is true for $a < 1/2$ also (see Section 4.6.3 for a related result), but our proof is not applicable in this case, since it heavily depends on Proposition 4.3.9. To state and prove our result, we introduce the notation μ_N for the stationary distribution for the model on N sites, with expectation and variance $\mathbb{E}^N$ and Var^N, respectively.

Theorem 4.5.11. *In the $(N,[a,b])$ model with $a \geq \frac{1}{2}$, for the unique stationary measure μ_N we have*

$$\lim_{N\to\infty} \mu_N = \delta_{\mathbb{E}U} \tag{4.5.12}$$

where $\delta_{\mathbb{E}U}$ denotes the Dirac measure concentrating on the (infinite-volume) constant configuration $\eta_i = \mathbb{E}U$ for all $i \in \mathbb{N}$, and where the limit is in the sense of weak convergence of probability measures.

We will prove this theorem by showing that for η distributed according to μ_N, in the limit $N \to \infty$, for every sequence $1 \leq j_N \leq N$,

1. $\lim_{N\to\infty} \mathbb{E}^N \eta_{j_N} = \mathbb{E}U$,

2. $\lim_{N\to\infty} \mathrm{Var}^N(\eta_{j_N}) = 0$.

The proof of the first item is not difficult. However, the proof of the second part is complicated, and is split up into several lemma's.

Proof of Theorem 4.5.11, part (1). We choose as initial configuration $\eta \equiv \mathbf{0}$, the configuration with all N sites empty, so that according to Lemma 4.5.2, we can write

$$\eta_{j_N}(t) = \sum_{\theta=1}^{t} A_{\theta j_N}(t) U_\theta. \tag{4.5.13}$$

Denoting expectation for this process as $\mathbb{E}_{\mathbf{0}}^N$, we find, using Remark 4.5.4, that

$$\mathbb{E}_{\mathbf{0}}^N \eta_{j_N}(t) = \mathbb{E}U \ \mathbb{E}_{\mathbf{0}}^N \mathcal{R}(\eta_{j_N}(t)).$$

First, we take the limit $t \to \infty$. By Theorem 4.5.1, $\mathbb{E}_{\mathbf{0}}^N \eta_{j_N}(t)$ converges to $\mathbb{E}^N \eta_{j_N}$. From Proposition 4.3.9, it likewise follows that $\lim_{t\to\infty} \mathbb{E}_{\mathbf{0}}^N \mathcal{R}(\eta_j(t)) = \frac{N}{N+1}$. Inserting these and subsequently taking the limit $N \to \infty$ proves the first part. □

For the proof of the second, more complicated part, we need a number of lemma's. First, we rewrite $\mathrm{Var}^N(\eta_{j_N})$ in the following manner.

Lemma 4.5.14.

$$\mathrm{Var}^N(\eta_{j_N}) = \mathrm{Var}(U) \lim_{t\to\infty} \mathbb{E}_{\mathbf{0}}^N \left[\sum_{\theta=1}^{t} (A_{\theta j_N}(t))^2\right] + (\mathbb{E}U)^2 \frac{N}{(N+1)^2}.$$

Proof. We start from expression (4.5.13), and use that the corresponding variance $\mathrm{Var}_{\mathbf{0}}^N$ converges to the stationary Var^N as $t \to \infty$ by Theorem 4.5.1. We rewrite, for fixed N and $j_N = j$,

$$\begin{aligned}
\mathrm{Var}_{\mathbf{0}}^N(\eta_j(t)) &= \mathbb{E}_{\mathbf{0}}^N\left[(\eta_j(t))^2\right] - \left[\mathbb{E}_{\mathbf{0}}^N \eta_j(t)\right]^2 \\
&= \mathbb{E}_{\mathbf{0}}^N\left[\left(\sum_{\theta=1}^{t} A_{\theta j}(t) U_\theta\right)^2\right] - \left[\mathbb{E}_{\mathbf{0}}^N \sum_{\theta=1}^{t} A_{\theta j}(t) U_\theta\right]^2 \\
&= \mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t} (A_{\theta j}(t))^2 U_\theta^2 + \sum_{\theta\neq\theta'} A_{\theta j}(t) U_\theta A_{\theta' j}(t) U_{\theta'}\right] - \left[\mathbb{E}_{\mathbf{0}}^N \sum_{\theta=1}^{t} A_{\theta j}(t) U_\theta\right]^2 \\
&= \mathbb{E}(U^2)\mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t} (A_{\theta j}(t))^2\right] + (\mathbb{E}U)^2 \mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta\neq\theta'} A_{\theta j}(t) A_{\theta' j}(t)\right] - (\mathbb{E}U)^2\left[\mathbb{E}_{\mathbf{0}}^N \sum_{\theta=1}^{t} A_{\theta j}(t)\right]^2 \\
&= \left(\mathbb{E}(U^2) - (\mathbb{E}U)^2\right) \mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t} (A_{\theta j}(t))^2\right]
\end{aligned}$$

$$+(\mathbb{E}U)^2 \left[\mathbb{E}_{\mathbf{0}}^N \left(\sum_{\theta=1}^{t} A_{\theta j}(t) \right)^2 - \left(\mathbb{E}_{\mathbf{0}}^N \sum_{\theta=1}^{t} A_{\theta j}(t) \right)^2 \right]$$
$$= \mathrm{Var}(U)\ \mathbb{E}_{\mathbf{0}}^N \left[\sum_{\theta=1}^{t} (A_{\theta j}(t))^2 \right] + (\mathbb{E}U)^2 \mathrm{Var}_{\mathbf{0}}^N(\mathcal{R}(\eta_j(t))).$$

Where in the third equality we used the independence of the A-coefficients of the added amounts U_θ. We now insert $j = j_N$, take the limit $t \to \infty$, and insert $\lim_{t\to\infty} \mathrm{Var}_{\mathbf{0}}^N(\mathcal{R}(\eta_{j_N}(t))) = \mathrm{Var}^N(\mathcal{R}(\eta_{j_N})) = \frac{N}{(N+1)^2}$. □

Arrived at this point, in order to prove Theorem 4.5.11, it suffices to show that

$$\lim_{N\to\infty} \lim_{t\to\infty} \mathbb{E}_{\mathbf{0}}^N \left[\sum_{\theta=1}^{t} (A_{\theta j_N}(t))^2 \right] = 0. \tag{4.5.15}$$

The next lemma's are needed to obtain an estimate for this expectation. We will adopt the strategy of showing that the factors $A_{\theta j}(t)$ are typically small, so that the energy of a typical site consists of many tiny fractions of additions. To make this precise, we start with considering one fixed $\theta > N(N-1)$, a time $t > \theta$, and we fix $\epsilon > 0$.

Definition 4.5.16. *We say that the event $G_t(\alpha)$ occurs, if $\max_j A_{\theta j}(t) \leq \alpha$. We say that the event $H_t(\epsilon)$ occurs, if $\max_j A_{\theta j}(t) \geq \epsilon$, and if in addition there is a lattice interval of size at most $M = \lceil \frac{1}{\epsilon} \rceil + 1$, containing X_θ, such that for all sites j outside this interval, $A_{\theta j}(t) \leq \epsilon$. (We call the mentioned interval the θ-heavy interval.)*

Note that since we have $\sum_j A_{\theta j}(t) \leq 1$ for every θ, the number of sites where $A_{\theta j}(t) \geq \epsilon$, cannot exceed $\lceil \frac{1}{\epsilon} \rceil$. In Lemma 4.5.8, we proved that $\max_j A_{\theta j}(t)$ is nonincreasing in t, for $t \geq \theta$. Therefore, also $G_t(\alpha)$ is increasing in t. This is not true for $H_t(\epsilon)$, because after an avalanche, the sites where $A_{\theta j}(t) > \epsilon$ might not form an appropriate interval around X_θ.

In view of what we want to prove, the events $G_t(\epsilon)$ and $H_t(\epsilon)$ are good events, because they imply that (if we think of N as being much larger than M) $A_{\theta j}(t) \leq \epsilon$ "with large probability". In the case that $G_t(\epsilon)$ occurs, $A_{\theta i}(t) \leq \epsilon$ for all i, and in the case that $H_t(\epsilon)$ occurs, there can be sites that contain a large $A_{\theta i}(t)$, but these sites are in the θ-heavy interval containing X_θ. This latter random variable is uniformly distributed on $\{1, \ldots, N\}$, so that there is a large probability that a particular j does not happen to be among them. If we only know that $G_t(\alpha)$ occurs for some $\alpha > \epsilon$, then we cannot draw such a conclusion. However, we will see that this is rarely the case.

Lemma 4.5.17. *For every N, for every $\theta > N(N-1)$, for every $\epsilon > 0$, for every K and j,*

1. *there exists a constant $c = c(\epsilon)$, such that for $\theta \leq t \leq \theta + K$*

$$\mathbb{P}^N_{\mathbf{0}}(A_{\theta j}(t) > \epsilon) \leq \frac{cK}{N};$$

2. *for every N large enough, there exist constants $w = w(\epsilon)$ and $0 < \gamma = \gamma(N, \epsilon) < 1$, such that for $t > \theta$*

$$\mathbb{P}^N_{\mathbf{0}}(A_{\theta j}(t) > \epsilon) \leq (1-\gamma)^{t-\theta-3w}.$$

In the proof of Lemma 4.5.17, we need the following technical lemma.

Lemma 4.5.18. *Consider a collection of real numbers $y_i \geq 0$, indexed by $\mathbb{N}$, with $\sum_i y_i \leq 1$ and such that for some $x \in \mathbb{N}$, $\max_{i \neq x} y_i \leq \alpha$. Then, for $j \geq x + \lceil \frac{1}{\alpha} \rceil$, we have*

$$\sum_{i=1}^{j-x+2} \frac{1}{2^i} y_{j-i+2} \;\leq\; f(\alpha) := \left(1 - \frac{1}{2^{\lceil \frac{1}{\alpha} \rceil}}\right)\alpha.$$

Proof. We write

$$\begin{aligned}
\sum_{i=1}^{j-x+2} \frac{1}{2^i} y_{j-i+2} &= \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i} y_{j-i+2} + \sum_{i=\lceil \frac{1}{\alpha} \rceil+1}^{j-x+2} \frac{1}{2^i} y_{j-i+2} \\
&\leq \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i} y_{j-i+2} + \frac{1}{2^{\lceil \frac{1}{\alpha} \rceil+1}} \sum_{i=\lceil \frac{1}{\alpha} \rceil+1}^{j-x+2} y_{j-i+2}.
\end{aligned} \tag{4.5.19}$$

Note that index x is in the second sum. For $i = 1, \ldots, \lceil \frac{1}{\alpha} \rceil$, write

$$y_{j-i+2} = \alpha - z_i,$$

with $0 \leq z_i \leq \alpha$. Then, since $\alpha \lceil \frac{1}{\alpha} \rceil \geq 1$ and $\sum_i y_i \leq 1$, we have $\sum_{i=\lceil \frac{1}{\alpha} \rceil+1}^{j-x+2} y_{j-i+2} \leq \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} z_i$, so that

$$\sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i} y_{j-i+2} + \frac{1}{2^{\lceil \frac{1}{\alpha} \rceil+1}} \sum_{i=\lceil \frac{1}{\alpha} \rceil+1}^{j-x+2} y_{j-i+2} \leq \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i}(\alpha - z_i) + \frac{1}{2^{\lceil \frac{1}{\alpha} \rceil+1}} \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} z_i$$

$$= \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i}\alpha - \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \left(\frac{1}{2^i} - \frac{1}{2^{\lceil \frac{1}{\alpha} \rceil + 1}} \right) z_i \leq \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i}\alpha,$$

where in the last step we used $z_i \geq 0$. Thus

$$\sum_{i=1}^{j-x+2} \frac{1}{2^i} y_{j-i+2} \leq \sum_{i=1}^{\lceil \frac{1}{\alpha} \rceil} \frac{1}{2^i}\alpha = \left(1 - \frac{1}{2^{\lceil \frac{1}{\alpha} \rceil}}\right)\alpha.$$

□

Proof of Lemma 4.5.17, part (1). We first discuss the case $t = \theta$. We show that $H_\theta(\epsilon)$ occurs, for arbitrary ϵ.

At $t = \theta$, the addition is made at X_θ. From Proposition 4.3.12 part (3), it follows, for every ϵ, that if after an avalanche there are sites j with $A_{\theta j}(\theta) > \epsilon$ (we will call such sites "θ-heavy" sites), then these sites form a set of adjacent sites including X_θ, except for a possible empty site among them. Since we have $\sum_{j=1}^{N} A_{\theta j}(\theta) \leq 1$, there can be at most $\lceil \frac{1}{\epsilon} \rceil$ θ-heavy sites. If the addition was made to an empty site, then $A_{\theta X_\theta}(\theta) = 1$. Thus, the θ-heavy interval has length at most $1 + \lceil \frac{1}{\epsilon} \rceil$, and we conclude that $H_\theta(\epsilon)$ occurs. To estimate the probability that $A_{\theta j}(\theta) > \epsilon$, or in other words, the probability that a given site j is in the θ-heavy interval, we use that X_θ is uniformly distributed on $\{1, \ldots, N\}$. Site j can be in the $\theta-$heavy interval if the distance between X_θ and j is at most $1 + \lceil \frac{1}{\epsilon} \rceil$, so that $\mathbb{P}_{\mathbf{0}}^N(A_{\theta j}(\theta) > \epsilon) \leq 2\frac{1+\lceil \frac{1}{\epsilon} \rceil}{N} =: \frac{c_2(\epsilon)}{N}$.

We next discuss $\theta < t \leq \theta + K$. We introduce the following constants. We choose a number w such that $f^w(1) \leq \epsilon$, with f as in Lemma 4.5.18, and where f^w denotes f composed with itself w times. Note that this is possible because $\lim_{k\to\infty} f^k(1) = 0$. We choose a combination of $\tilde{\epsilon}_0$ and d such that $\tilde{\epsilon}_w \leq \epsilon$, with $\tilde{\epsilon}_{k+1}$ defined as $\tilde{\epsilon}_k + \frac{1}{2^{d+1}}(f^k(1) - \tilde{\epsilon}_k)$. Finally, we define $\tilde{M}_k = \lceil \frac{1}{\tilde{\epsilon}_0} \rceil + 1 + k(1+d)$.

For fixed θ and a time $t > \theta$, we define three types of avalanches at time t: "good", "neutral" and "bad".

Definition 4.5.20. *For a fixed θ, the avalanche at time t is*

- *a* good *avalanche if the following conditions are satisfied:*
  1. *X_t and X_θ are on the same side of the empty site (if present) at $t-1$,*
  2. *X_θ is at distance at least $\tilde{M}_w$ from the boundary,*
  3. *X_t is at distance at least w from the boundary, and from the empty site (if present) at $t-1$,*
  4. *X_t is at distance at least $\tilde{M}_w + \lceil \frac{1}{\epsilon} \rceil$ from X_θ,*

5. X_θ is at distance at least $\tilde{M}_w$ from the empty site (if present) at $t-1$,

- *a* neutral *avalanche if condition (5) is satisfied, but (1) is not,*
- *a* bad *avalanche in all other cases.*

Having defined the three kinds of avalanches, we now claim the following:

- If $H_{t-1}(\epsilon)$ occurs, then after a neutral avalanche, $H_t(\epsilon)$ occurs.
- If $H_{t-1}(\epsilon)$ occurs, then after a good avalanche, $G_t(\epsilon)$ occurs.

The first claim (about the neutral avalanche) holds because if condition (5) is satisfied, but (1) is not, then not only is X_θ not in the range of the avalanche, but the distance of X_θ to the empty site is large enough to guarantee that the entire θ-heavy interval is not in the range. Thus, the θ-heavy interval does not topple in the avalanche. It then automatically follows that $H_t(\epsilon)$ occurs.

To show the second claim (about the good avalanche), more work is required. We break the avalanche up into waves. Using a similar notation as in the proof of Proposition 4.3.12, we will denote by $\tilde{A}_{\theta j}(k)$ the fraction of U_θ at site j after wave k. We also define another event: we say that $\tilde{H}_k(\tilde{M}_k, \tilde{\alpha}_k, \tilde{\epsilon}_k)$ occurs, if $\max_j \tilde{A}_{\theta j}(k) \leq \tilde{\alpha}_k$, and all sites where $\tilde{A}_{\theta j}(k) > \epsilon$ are in an interval of length at most $\tilde{M}_k$ containing X_θ (we will call this the (k,θ)-heavy interval), with the exception of site X_t when it is unstable, in which case we require that $\tilde{A}_{\theta X_t}(k) \leq 2\tilde{\epsilon}_k$.

We define a "good" wave, as a wave in which all sites of the (k,θ)-heavy interval topple, and the starting site X_t is at a distance of at least $\frac{1}{\alpha_k}$ from the (k,θ)-heavy interval. It might become clear now that Definition 4.5.20 has been designed precisely so that a good avalanche is an avalanche that starts with at least w good waves. We will now show by induction in the number of waves that after an avalanche that starts with w good waves, $G_t(\epsilon)$ occurs.

For $k = 0$, we choose $\tilde{\alpha}_0 = 1$, so that at $k = 0$, $\tilde{H}_0(\tilde{M}_0, \tilde{\alpha}_0, \tilde{\epsilon}_0)$ occurs. We will choose $\tilde{\alpha}_{k+1} = f(\tilde{\alpha}_k)$ and $\tilde{\epsilon}_{k+1} = \tilde{\epsilon}_k + \frac{1}{2^{d+1}}(\tilde{\alpha}_k - \tilde{\epsilon}_k)$, so that once $\tilde{H}_w(\tilde{M}_w, \tilde{\alpha}_w, \tilde{\epsilon}_w)$ occurred, we are sure that after the avalanche $G_t(\epsilon)$ occurs, because both $\tilde{\alpha}_w$ and $\tilde{\epsilon}_w$ are smaller than ϵ. Now all we need to show is that, if $\tilde{H}_k(\tilde{M}_k, \tilde{\alpha}_k, \tilde{\epsilon}_k)$ occurs, then after a good wave $\tilde{H}_{k+1}(\tilde{M}_{k+1}, \tilde{\alpha}_{k+1}, \tilde{\epsilon}_{k+1})$ occurs.

From (4.3.15), we see that, for all $j > X_t$ that topple (and do not become empty),

$$\tilde{A}_{\theta j}(k+1) = \frac{1}{2}\tilde{A}_{\theta,j+1}(k) + \frac{1}{4}\tilde{A}_{\theta j}(k) + \frac{1}{8}\tilde{A}_{\theta,j-1}(k) + \cdots + \frac{1}{2^{j-X_t+2}}\tilde{A}_{\theta,X_t}(k), \quad (4.5.21)$$

and similarly for $j < X_t$. For $j = X_t$, we have

$$\begin{aligned}\tilde{A}_{\theta j}(k+1) \quad &= \quad \left(\frac{1}{2}\tilde{A}_{\theta,j+1}(k) + \frac{1}{4}\tilde{A}_{j,\theta}(k)\right)\mathbf{1}_{\tilde{A}_{\theta,j+1}(k)\neq 0} \\ &\quad + \left(\frac{1}{2}\tilde{A}_{\theta,j-1}(k) + \frac{1}{4}\tilde{A}_{j,\theta}(k)\right)\mathbf{1}_{\tilde{A}_{\theta,j-1}(k)\neq 0}.\end{aligned} \tag{4.5.22}$$

First we use that in a good wave, all sites in the θ-heavy interval topple, and X_t is not in this interval. We denote by m the leftmost site of the θ-heavy interval, so that the rightmost site is $m + \tilde{M}(k)$. Suppose, without loss of generality, that $X_t < m$. We substitute $\tilde{A}_{\theta j}(k) \leq \tilde{\alpha}_k$ for all j in the θ-heavy interval, $\tilde{A}_{X_t\theta}(k) \leq 2\tilde{\epsilon}_k$, and $\tilde{A}_{\theta j}(k) \leq \tilde{\epsilon}_k$ otherwise into (4.5.21), to derive for all j that topple:

$$\begin{array}{lll} j < m-1, j \neq x & \tilde{A}_{\theta j}(k+1) \leq \tilde{\epsilon}_k, & \\ j = m-1, \ldots, m+\tilde{M}(k) & \tilde{A}_{\theta j}(k+1) < \tilde{\alpha}_k, & \\ j = m + \tilde{M}(k) + d' & \tilde{A}_{\theta j}(k+1) < \tilde{\epsilon}_k + \frac{1}{2^{d'+1}}(\tilde{\alpha}_k - \tilde{\epsilon}_k), & d' = 1, 2, \ldots \end{array} \tag{4.5.23}$$

Additionally, by (4.5.22), we have $\tilde{A}_{\theta,X_t}(k+1) \leq 2\tilde{\epsilon}_k$. The factor 2 is only there as long as site X_t is unstable. From (4.5.23) we have that $\tilde{\alpha}_{k+1} < \tilde{\alpha}_k$, but moreover, in a good wave, the variables $\tilde{A}_{\theta j}(k)$ satisfy the conditions of Lemma 4.5.18, so that in fact $\tilde{\alpha}_{k+1} \leq f(\tilde{\alpha}_k)$. If we insert our choice of d for d', then we can see from (4.5.23) that indeed after the good wave $\tilde{H}_{k+1}(\tilde{M}_{k+1}, \tilde{\alpha}_{k+1}, \tilde{\epsilon}_{k+1})$ occurs.

Now we are ready to evaluate $\mathbb{P}^N_{\mathbf{0}}(A_{\theta j}(t) > \epsilon)$, for $t \in \{\theta+1, \ldots, \theta+K\}$. As is clear by now, there are three possibilities: $G_t(\epsilon)$ occurs, so that $\max_j A_{\theta j}(t) \leq \epsilon$, or $H_t(\epsilon)$ occurs, in which case $A_{\theta j}(t)$ can be larger than ϵ if j is in the θ-heavy interval. We derived in the case $t = \theta$ that the probability for this is bounded above by $\frac{c_2}{N}$. Finally, it is possible that neither occurs, in which case we do not have an estimate for the probability that $A_{\theta j}(t) > \epsilon$. But for this last case, we must have had at least one bad avalanche between $\theta + 1$ and t. We will now show that the probability of this event is bounded above by $\frac{Kc_1}{N}$, where c_1 depends only on ϵ.

As stated in Definition 4.5.20, a bad avalanche can occur at time t if at least one of the conditions (2) through (5) is not satisfied. Thus, we can bound the total probability of a bad avalanche at time t, by summing the probabilities that the various conditions are not satisfied. We discuss the conditions one by one.

- The probability that condition (2) is not satisfied, is bounded above by $\frac{2\tilde{M}_w}{N}$, since X_θ is distributed uniformly on $\{1, \ldots, N\}$.
- The probability that condition (3) is not satisfied, is bounded above by $\frac{4w}{N}$, since X_t is distributed uniformly on $\{1, \ldots, N\}$, and independent of the position of the empty site at $t-1$, if present.

- The probability that condition (4) is not satisfied, is bounded above by $\frac{2(\tilde{M}_w+\lceil\frac{1}{\epsilon}\rceil)}{N}$, since X_t and X_θ are independent.
- The probability that condition (5) is not satisfied is bounded above by $\frac{2\tilde{M}_w}{N-1}$, since the position of the empty site at $t-1$ is almost uniform on $\{1,\dots,N\}$ (recall the notion of almost uniformity, mentioned after Proposition 4.3.9).

Thus, the total probability of a bad avalanche at time t is bounded by $\frac{2\tilde{M}_w}{N} + \frac{4w}{N} + \frac{2(\tilde{M}_w+\lceil\frac{1}{\epsilon}\rceil)}{N} + \frac{2\tilde{M}_w}{N} \equiv \frac{c_1}{N}$, so that the probability of at least one bad avalanche between $\theta+1$ and t is bounded by $\frac{Kc_1}{N}$. We conclude that for $t \in \{\theta+1,\dots,\theta+K\}$, $\mathbb{P}^N_{\mathbf{0}}(A_{\theta j}(t) > \epsilon) \le \frac{Kc_1+c_2}{N} \le \frac{cK}{N}$, for some $c > 0$. □

Proof of Lemma 4.5.17, part (2). From Lemma 4.5.18, it follows that if $G_t(\alpha)$ occurs, and after s time steps all θ-heavy sites have toppled at least once, in avalanches that all start at least a distance $\lceil\frac{1}{\alpha}\rceil$ from all current θ-heavy sites, then $G_{t+s}(f(\alpha))$ occurs; we will exploit this fact.

Suppose that $G_t(\alpha)$ occurs and that in addition, at time t the empty set is almost uniform. We claim that this implies that $G_{t+2}(f(\alpha))$ occurs with a probability that is bounded below, uniformly in N and ϵ. To see this, observe that if there is no empty site, then all N sites topple in one time step. If there is an empty site, this (meaning all sites topple) also happens in two steps if in the first step, we add to one side of the empty site, and in the second step to the other side. Denote by e_1 the position of the empty site before the first addition (at X_{t+1}), and e_2 before the second addition (at X_{t+2}). If $X_{t+1} < e_1$, then $e_2 < e_1$. Therefore, all sites topple if $X_{t+1} < e_1$ and $X_{t+2} > e_1$. With the distribution of e_1 uniform on $\{1,\dots,N\}$, the probability that this happens is bounded below by some constant γ' independent of N and ϵ. However we have the extra demand that both additions should start at least a distance $\lceil\frac{1}{\alpha}\rceil \le \lceil\frac{1}{\epsilon}\rceil$ from all current θ-heavy sites, of which there are at most $\lceil\frac{1}{\epsilon}\rceil$. Thus, both additions should avoid at most $\lceil\frac{1}{\epsilon}\rceil^2$ sites. The probability that this happens is therefore less than some $\gamma' > 0$, but it is easy to see that the difference decreases with N. We can then conclude that there is an N' large enough so that the probability that this happens is at least $\gamma > 0$ for all $N \ge N'$, with $0 < \gamma < 1$ independent of N and ϵ.

In view of this, the probability that $G_{\theta+2}(f(1))$ occurs, for N large enough, is at least γ. We wish to iterate this argument w times. However, the lower bound γ is only valid when the empty site is almost uniform on $\{1,\dots,N\}$. We do not have this for $\eta(\theta+2)$ since we have information about what happened in the time interval $(\theta,\theta+2)$. However, after one more addition at an unknown position, the empty site in $\eta(\theta+3)$ is again almost uniform on $\{1,\dots,N\}$. Since $f^w(1) \le \epsilon$, iterating this argument gives

$$\mathbb{P}^N_{\mathbf{0}}(\max_j A_{\theta j}(t) > \epsilon) \le (1-\gamma)^{t-\theta-3w}.$$

□

Proof of Theorem 4.5.11, part (2). By Lemma 4.5.14, it suffices to prove (4.5.15). We estimate, using that $\sum_\theta A_{\theta j_N}(t) \leq 1$,

$$\mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t}(A_{\theta j_N}(t))^2\right] \leq \mathbb{E}_{\mathbf{0}}^N\left[\max_{1\leq\theta\leq t} A_{\theta j_N}(t)\sum_{\theta=1}^{t} A_{\theta j_N}(t)\right] \leq \mathbb{E}_{\mathbf{0}}^N\left[\max_{1\leq\theta\leq t} A_{\theta j_N}(t)\right]$$

$$\leq \mathbb{E}_{\mathbf{0}}^N\left[\max_{t-K\leq\theta\leq t} A_{\theta j_N}(t)\right]+\mathbb{E}_{\mathbf{0}}^N\left[\max_{N(N+1)<\theta<t-K} A_{\theta j_N}(t)\right]+\mathbb{E}_{\mathbf{0}}^N\left[\max_{1\leq\theta\leq N(N+1)} A_{\theta j_N}(t)\right].$$

We then for the first two terms estimate, using that $\max_\theta A_{\theta j_N}(t) \leq 1$,

$$\mathbb{E}_{\mathbf{0}}^N[\max_\theta A_{\theta j_N}(t)] \leq \epsilon + \mathbb{P}_{\mathbf{0}}^N(\max_\theta A_{\theta j_N}(t) > \epsilon) \leq \epsilon + \sum_\theta \mathbb{P}_{\mathbf{0}}^N(A_{\theta j_N}(t) > \epsilon).$$

We finally use Lemma 4.5.17, and choose $K = K_N$ increasing with N. For $\theta \in [t, t-K_N]$, we straightforwardly obtain $\sum_{\theta=t-K_N}^{t} \mathbb{P}_{\mathbf{0}}^N\,(A_{\theta j_N}(t) > \epsilon) = O(\frac{K_N^2}{N})$, uniformly in t, as $N \to \infty$. For $\theta < t - K_N$ we calculate

$$\sum_{\theta<t-K_N} \mathbb{P}_{\mathbf{0}}^N\,(A_{\theta j_N}(t) > \epsilon) \leq \sum_{t-\theta>K_N}(1-\gamma)^{t-\theta-3w} = O((1-\gamma)^{K_N})$$

as $N \to \infty$, so that

$$\mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t}(A_{\theta j_N}(t))^2\right] \leq 2\epsilon + O(\frac{K_N^2}{N}) + O((1-\gamma)^{K_N}) + \mathbb{E}_{\mathbf{0}}^N\left[\max_{1\leq\theta\leq N(N+1)} A_{\theta j_N}(t)\right],$$

as $N \to \infty$. In the limit $t \to \infty$, by Lemma 4.5.8 part (2), the last term vanishes. We now choose $K_N = N^{1/3}$, to obtain

$$\limsup_{N\to\infty}\lim_{t\to\infty} \mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t}(A_{\theta j_N}(t))^2\right] \leq 2\epsilon,$$

Since $\epsilon > 0$ is arbitrary, we finally conclude that

$$\lim_{N\to\infty}\lim_{t\to\infty} \mathbb{E}_{\mathbf{0}}^N\left[\sum_{\theta=1}^{t}(A_{\theta j_N}(t))^2\right] = 0.$$

□

4.6 The $(N, [0,1])$-model

4.6.1 Uniqueness of the stationary distribution

Theorem 4.6.1. *The* $(N, [0,1])$ *model has a unique stationary distribution* υ_N*. For every initial distribution* ν *on* Ω_N*,* $\mathbb{P}_\nu$ *converges in total variation to* υ_N*.*

Proof. We prove this theorem again by constructing a successful coupling. For clarity, we first treat the case $N = 2$, and then generalize to $N > 2$. The coupling is best described in words.

Using the same notation as in previous couplings, we call two independent copies of the process $\eta^1(t)$ and $\eta^2(t)$, and call the coupled processes $\hat{\eta}^1(t)$ and $\hat{\eta}^2(t)$. Initially, we choose $\hat{\eta}^1(t) = \eta^1(t)$, and $\hat{\eta}^2(t) = \eta^2(t)$. It is easy, but tedious, to show that $\eta_1^1(t) = \eta_1^2(t) = 0$, while $\mathcal{R}(\eta_2^1(t)) = \mathcal{R}(\eta_2^2(t)) = 1$, occurs infinitely often.

At the first such time T_1 that this occurs, we choose the next addition as follows. Call $\Delta(t) = \eta_2^1(t) - \eta_2^2(t)$. We choose $\hat{X}^2_{T_1+1} = X^1_{T_1+1}$, and $\hat{U}^2_{T_1+1} = (U^1_{T_1+1} + \Delta(T_1)) \bmod 1$. Observe that the distribution of $\hat{U}^2_{T_1+1}$ is uniform on $[0,1]$.

This addition is such that with positive probability the full sites are chosen for the addition, and the difference $\Delta(T_1)$ is canceled. More precisely, this occurs if $X^1_{T_1+1} = 2$, which has probability $1/2$, and $(U^1_{T_1+1} + \Delta(T_1)) \bmod 1 = U^1_{T_1+1} + \Delta(T_1)$, which has probability at least $1/2$, since $\eta_2^1(T_1)$ and $\eta_2^2(T_1)$ are both full, therefore $\Delta(T_1) \leq 1/2$. If this occurs, then we achieve success, i.e., $\hat{\eta}^1(T_1+1) = \hat{\eta}^2(T_1+1)$, and from that time on we can let the two coupled processes evolve together.

If $\hat{\eta}^1(T_1+1) \neq \hat{\eta}^2(T_1+1)$, then we evolve the two coupled processes independently, and repeat the above procedure at the next instant that $\hat{\eta}_1^1(t) = \hat{\eta}_1^2(t) = 0$. Since at every such instant, the probability of success is positive, we only need a finite number of attempts. Therefore, the above constructed coupling is successful, and this proves the claim for $N = 2$.

We now describe the coupling in the case $N > 2$. We will again evolve two processes independently, until a time where $\eta_1^1(t) = \eta_1^2(t) = 0$, while all other sites are full. At this time we will attempt to cancel the differences on the other $N-1$ sites one by one. We define $\Delta_j(t) = \eta_j^1(t) - \eta_j^2(t)$, and as before we would be successful if we could cancel all these differences. However, now that $N > 2$, we do not want an avalanche to occur during this equalizing procedure, because we need $\eta_1^1(t) = \eta_1^2(t) = 0$ during the entire procedure. Therefore, we specify T_1 further: T_1 is the first time where not only $\eta_1^1(t) = \eta_1^2(t) = 0$ and all other sites are full, but also $\eta_j^1(t) < 1 - \epsilon$ and $\eta_j^2(t) < 1 - \epsilon$, for all $j = 2, \ldots, N$, with $\epsilon = \frac{1}{2^{N+1}}$. At such a time, a positive amount can be added to each site without starting an avalanche. We will first show that this occurs infinitely often, which also settles the case $N = 2$.

By Proposition 4.3.4, after a finite time $\eta^1(t)$ and $\eta^2(t)$ contain at most one non-full site. It now suffices to show that for any $\xi(t) \in \Omega_N$ with at most one non-full site, with positive probability the event that $\xi_1(t+4) = 0$, while $\xi_j(t+4) \leq 1 - \frac{1}{2^{N+1}}$ for every $2 \leq j \leq N$, occurs.

One explicit possibility is as follows. The first addition should cause an avalanche. This will ensure that $\xi(t+1)$ contains one empty site. This occurs if the addition site is a full site, and the addition is at least $1/2$. The probability of this is at least $\frac{1}{2}(1 - \frac{1}{N})$. The second addition should change the empty site into full. For this to occur, the addition should be at least $1/2$, and the empty site should be chosen. This has probability $\frac{1}{2N}$. The third addition should be at least $1/2$ to site 1, so that an avalanche is started that will result in $\xi_N(t+3) = 0$. This has again probability $\frac{1}{2N}$. Finally, the last addition should be an amount in $[\frac{1}{2}, \frac{3}{4}]$, to site $N-1$. Then by (4.3.15), every site but site N will topple once, and after this avalanche, site 1 will be empty, while every other site contains at most $1 - \frac{1}{2^{N+1}}$. This last addition has probability $\frac{1}{4N}$.

Now we show that at time T_1 defined as above, there is a positive probability of success. To choose all full sites one by one, we require, first, for all $j = 2, \ldots, N$ that $X^1_{T_1+j-1} = j$. This has probability $(\frac{1}{N})^{N-1}$. Second, we need $(U^1_{T_1+j-1} + \Delta_j(T_1)) \bmod 1 = U^1_{T_1+j-1} + \Delta_j(T_1)$ for all $j = 2, \ldots, N$. This event is independent of the previous event and has probability at least $(\frac{1}{2})^{N-1}$. If this second condition is met, then third, we need to avoid avalanches, so for all $j = 2, \ldots, N$, $\eta^1_j(T_1 + j - 1) + U^1_{T_1+j-1} = \eta^2_j(T_1 + j - 1) + \hat{U}^2_{T_1+j-1} < 1$. It is not hard to see that this has positive conditional probability, given the previous events. We conclude that the probability of success at time $T_1 + N - 1$ is positive, so that we only need a finite number of such attempts. Therefore, the coupling is successful, and we are done. □

4.6.2 Simulations

We performed Monte Carlo simulations of the $(N,[0,1])$-model, for various values of N. Figure 4.3 shows histograms of the energies that a site assumes during all the iterations. We started from the empty configuration, but omitted the first 10% of the observations to avoid recording transient behavior. Further increasing this percentage, or the number of iterations, had no visible influence on the results.

The presented results show that, as the number of sites of the model increases, the energy becomes more and more concentrated around a value close to 0.7. In the next section, we present an argument for this value to be $\sqrt{1/2}$. We further observe that it seems to make a difference where the site is located: at the boundary the variance seems to be larger than in the middle.

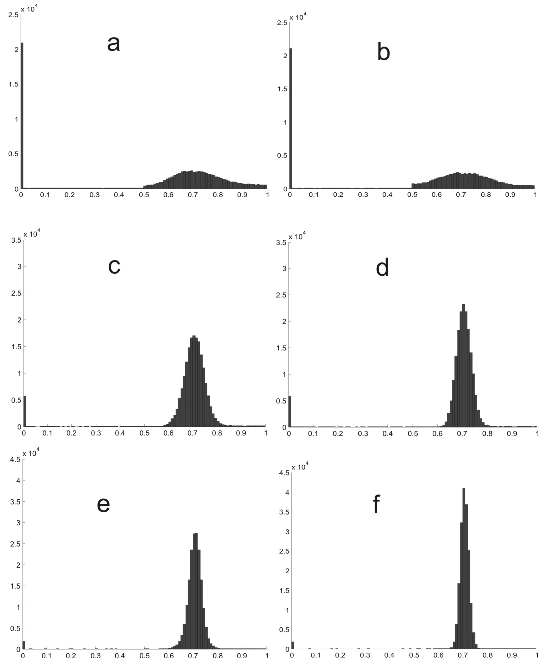

Figure 4.3: Simulation results for the $(N, [0, 1])$-model. The histograms represent observed energies during 100,000 (a,b) and 200,000 iterations (c-f). The system size was 3 sites (a,b), 30 sites (c,d) and 100 sites (e,f). (a),(c) and (e) are boundary sites, (b), (d) and (f) are central sites.

4.6.3 The expected stationary energy per site as $N \to \infty$

From the simulations it appears that for large values of N, the energy per site concentrates at a value close to 0.7, for every site. Below we argue, under some assumptions that are consistent with our simulations, that this value should be $\sqrt{1/2}$.

First, we assume that every site has the same expected stationary energy. Moreover, we assume that pairs of sites are asymptotically independent, i.e., η_x becomes independent of η_y as $|x-y| \to \infty$. (If the stationary measure is indeed such that the energy of every site is a.s. equal to a constant, then this second assumption is clearly true.) With $\mathbb{E}_{\upsilon_N}$ denoting expectation with respect to the stationary distribution υ_N, we say that $(\upsilon_N)_N$ is *asymptotically independent* if for any $1 \leq x_N, y_N \leq N$ with $|x_N - y_N| \to \infty$, and for any A, B subsets of $\mathbb{R}$ with positive Lebesgue measure, we have

$$\lim_{N\to\infty} \Big(\mathbb{E}_{\upsilon_N}(\mathbf{1}_{\eta_{x_N} \in B} | \eta_{y_N} \in A) - \mathbb{E}_{\upsilon_N}(\mathbf{1}_{\eta_{x_N} \in B} \Big) = 0. \tag{4.6.2}$$

Theorem 4.6.3. *Suppose that in the $(N,[0,1])$ model, for any sequence $j_N \in \{1,\ldots,N\}$,*

$$\lim_{N\to\infty} \mathbb{E}_{\upsilon_N}(\eta_{j_N}) = \rho, \tag{4.6.4}$$

for some constant ρ. Suppose in addition that $(\upsilon_N)_N$ is asymptotically independent. Then we have $\rho = \sqrt{\frac{1}{2}}$.

Proof. The proof is based on a conservation argument. If we pick a configuration according to υ_N and we make an addition U, we denote the random amount that leaves the system by $E_{out,N}$. By stationarity, the expectation of U must be the same as the expectation of $E_{out,N}$.

The amount of energy that leaves the system in case of an avalanche, depends on whether or not one of the sites is empty (or behaves as empty). Remember (Proposition 4.3.4) that when we pick a configuration according to the stationary distribution, then there can be at most one empty or anomalous site. If there is one empty site, then the avalanche reaches one boundary. If there are only full sites, then the avalanche reaches both boundaries, and in case of one anomalous site, both can happen.

However, configurations with no empty site have vanishing probability as $N \to \infty$: we claim that the stationary probability for a configuration to have no empty site, is bounded above by p_N, with $\lim_{N\to\infty} p_N = 0$. To see this, we divide the support of the stationary distribution into two sets: $\mathcal{E}$, the set of configurations with one empty site, and $\mathcal{N}$, the set of configurations with no empty site. The only way to reach $\mathcal{N}$ from $\mathcal{E}$, is to make an addition precisely at the empty site. As X is uniformly distributed on $\{1,\ldots,N\}$, this has probability $\frac{1}{N}$, irrespective of the details of the configuration. The only way to reach $\mathcal{E}$ from $\mathcal{N}$, is to cause an avalanche; this certainly happens if an addition of at least $1/2$ is made to a full site. Again, since X is uniformly distributed on $\{1,\ldots,N\}$, and since there is at most one non-full site, this has probability at least $\frac{1}{2}\frac{N-1}{N}$.

Now let X be the (random) addition site at a given time, and denote by A_x the event that $X = x$ and that this addition causes the start of an avalanche. Since $E_{out,N} = 0$ when no avalanche is started, we can write

$$\mathbb{E}_{\upsilon_N}(E_{out,N}) = \sum_{x=1}^{N} \mathbb{E}_{\upsilon_N}(E_{out,N}|A_x)\mathbb{P}_{\upsilon_N}(A_x). \tag{4.6.5}$$

We calculate $\mathbb{P}_{\upsilon_N}(A_x)$ as follows, writing U for the value of the addition:

$$\begin{aligned}
\mathbb{P}_{\upsilon_N}(A_x) &= \frac{1}{N}\mathbb{P}_{\upsilon_N}(\eta_x + U \geq 1) = \frac{1}{N}\mathbb{P}_{\upsilon_N}(U \geq 1-\eta_x) \\
&= \frac{1}{N}\int \mathbb{P}_{\upsilon_N}(U \geq 1-\eta_x)d\upsilon_N(\eta) = \frac{1}{N}\int \eta_x d\upsilon_N(\eta) \\
&= \frac{1}{N}\mathbb{E}_{\upsilon_N}(\eta_x).
\end{aligned} \tag{4.6.6}$$

Let $L_N = \lceil \log N \rceil$. Even if the avalanche reaches both boundary sites, the amount of energy that leaves the system can never exceed 2, which implies that

$$\left| \sum_{x=1}^{N} \mathbb{E}_{\upsilon_N}(E_{out,N}|A_x) - \sum_{x=2L_N}^{N-2L_N} \mathbb{E}_{\upsilon_N}(E_{out,N}|A_x) \right| \le 8L_N. \tag{4.6.7}$$

It follows from (4.6.5), (4.6.6) and (4.6.7) that

$$\mathbb{E}_{\upsilon_N}(E_{out,N}) = \frac{1}{N} \sum_{x=2L_N}^{N-2L_N} \mathbb{E}_{\upsilon_N}(E_{out,N}|A_x)\mathbb{E}_{\upsilon_N}(\eta_x) + O(L_N/N). \tag{4.6.8}$$

If the avalanche, started at site x, reaches the boundary at site 1, then the amount of energy that leaves the system is given by $\frac{1}{2}\eta_1 + \frac{1}{4}\eta_2 + \cdots + (\frac{1}{2})^x(\eta_x + U)$. For all $x \in \{2L_N, \ldots, N-2L_N\}$, this can be written as

$$\frac{1}{2}\eta_1 + \frac{1}{4}\eta_2 + \cdots + (\frac{1}{2})^{L_N}\eta_{L_N} + (\frac{1}{2})^{L_N+1}\eta_{L_N+1} + \cdots + (\frac{1}{2})^x(\eta_x + U),$$

where for the last part of this expression, we have the bound

$$(\frac{1}{2})^{L_N+1}\eta_{L_N+1} + \cdots + (\frac{1}{2})^x(\eta_x + U) \le (\frac{1}{2})^{L_N}.$$

Since the occurrence of A_x depends only on η_x (and on X and U), for $2L_N \le x \le N - 2L_N$, by asymptotic independence there is an α_N, with $\lim_{N\to\infty} \alpha_N = 0$, such that for all $1 \le i \le L_N$ and $2L_N \le x \le N - 2L_N$, we have

$$|\mathbb{E}_{\upsilon_N}(\eta_i|A_x) - \mathbb{E}_{\upsilon_N}(\eta_i)| \le \alpha_N,$$

so that $\left|\mathbb{E}_{\upsilon_N}\left(\frac{1}{2}\eta_1 + \cdots + (\frac{1}{2})^{L_N}\eta_{L_N}|A_x\right) - \mathbb{E}_{\upsilon_N}\left(\frac{1}{2}\eta_1 + \cdots + (\frac{1}{2})^{L_N}\eta_{L_N}\right)\right|$
$\le \left(\frac{1}{2} + \frac{1}{4} + \cdots\right)\alpha_N$,
which is bounded above by α_N. By symmetry, we have a similar result in the case that the other boundary is reached. In case both boundaries are reached, we simply use that the amount of energy that leaves the system is bounded above by 2.

In view of this, we continue the bound in (4.6.8) as follows:

$$\begin{aligned}
&\mathbb{E}_{\upsilon_N}(E_{out,N}) = \\
= \quad &\frac{1}{N}\sum_{x=2L_N}^{N-2L_N}\left(\mathbb{E}_{\upsilon_N}(\frac{1}{2}\eta_1 + \frac{1}{4}\eta_2 + \cdots + (\frac{1}{2})^{L_N}\eta_{L_N}|A_x) + O((\frac{1}{2})^{L_N})\right)\mathbb{E}_{\upsilon_N}(\eta_x) \\
&+O(L_N/N) \\
= \quad &\frac{1}{N}\sum_{x=2L_N}^{N-2L_N}\mathbb{E}_{\upsilon_N}(\frac{1}{2}\eta_1 + \frac{1}{4}\eta_2 + \cdots + (\frac{1}{2})^{L_N}\eta_{L_N})\mathbb{E}_{\upsilon_N}(\eta_x) \\
&+O(\frac{L_N}{N}) + O((\frac{1}{2})^{L_N}) + O(\alpha_N) + O(p_N),
\end{aligned}$$

as $N \to \infty$. Letting $N \to \infty$ and inserting (4.6.4) now gives

$$\lim_{N\to\infty} \mathbb{E}_{\upsilon_N}(E_{out,N}) = \rho^2.$$

As the expectation of U is $\frac{1}{2}$, we conclude that $\rho = \sqrt{\frac{1}{2}}$. □

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Chapter 5

Limiting shapes

Reproduction of "Limiting shapes for deterministic centrally seeded growth models", by A. Fey and F. Redig [23]

abstract We study the rotor router model and two deterministic sandpile models. For the rotor router model in $\mathbb{Z}^d$, Levine and Peres proved that the limiting shape of the growth cluster is a sphere. For the other two models, only bounds in dimension 2 are known. A unified approach for these models with a new parameter h (the initial number of particles at each site), allows to prove a number of new limiting shape results in any dimension $d \geq 1$. For the rotor router model, the limiting shape is a sphere for all values of h. For one of the sandpile models, and $h = 2d-2$ (the maximal value), the limiting shape is a cube. For both sandpile models, the limiting shape is a sphere in the limit $h \to -\infty$. Finally, we prove that the rotor router shape contains a diamond.

5.1 Introduction

In a growth model, there is a dynamical rule by which vertices of a graph are added to an initial collection. The existing literature on growth models deals mainly with stochastic growth models. Among many stochastic growth models, there are for example the Eden model, the Richardson model, first and last passage percolation and diffusion limited aggregation. An introduction to stochastic growth models and limiting shape theorems can be found in [17].

An example related to the models in this paper, is the internal diffusion limited aggregation (IDLA) model. One adds particles one by one to the origin, letting each particle perform a random walk that stops when it hits an empty site. The growth cluster is then the collection of sites that contain a particle.

The IDLA model on $\mathbb{Z}^d$ has been studied by Lawler, Bramson and Griffeath. In their paper from 1992 [28], they prove that the limiting shape of the growth cluster for this model in any dimension is a sphere. Lawler [29] estimated the speed of convergence.

The three deterministic models to be discussed in this paper can be viewed as deterministic analogues of IDLA, and are all examples of a general height-arrow model studied in [9]. The closest analogue is the rotor router model, proposed by Propp (see [27]). Once more, particles are added to the origin of the d-dimensional lattice $\mathbb{Z}^d$, but now they perform a deterministic walk as follows: at each site, there is an arrow present, pointing at one of the $2d$ neighboring sites. If a particle at this site finds it already occupied, then it rotates the arrow to the next position, and takes a step in the new direction of the arrow.

In our other models, occupied sites hold the particles until they can be sent out in pairs, in the directions of a two-pointed arrow (double router model), or groups of $2d$, one in every direction (sandpile model). In the last model, exactly the same amount of particles is sent in every direction. In the other two models, the difference is at most one.

The last two models are deterministic versions of sandpile models. A stochastic sandpile model, where addition occurs at a random site, has originally been proposed to study self-organized criticality [5]. Soon after, Dhar [10] introduced the abelian sandpile model, that has the symmetric toppling rule ("toppling" of a site means that particles are sent out), which is now the most widely studied variant. Manna's sandpile model [37] is doubly stochastic: the addition site is random, and moreover, in a toppling two particles are sent to randomly chosen neighbors.

The rotor router model on $\mathbb{Z}^d$ has been studied by Peres and Levine [30, 31]. They have found that the limiting shape of this model is also a sphere, and give bounds for the rate of convergence. This result is at the basis of a number of new results in the present paper. Recently, a new paper of these authors appeared [32], further extending their results.

The deterministic abelian sandpile model has been studied on a finite square grid [34, 38, 50], with emphasis on the dynamics, but hardly as a growth model. Le Borgne and Rossin [7] found some bounds for the limiting shape for $d = 2$.

From the above introductory description, it should be clear that all the deterministic models we introduced here are closely related. In Section 5.2, we define the above models on $\mathbb{Z}^d$ in a common framework, to allow comparisons of the models. We introduce a parameter h to parametrize the initial configuration. For all models, a (large) number n of particles starts at the origin, and spreads out in a deterministic way through topplings. The rest of the grid initially contains a number h of particles at each site, where h can be negative. One can imagine a negative amount of particles as a hole that needs to be filled up; admitting negative particle numbers will be helpful in comparing the different models. Once every site is stable, (i.e., has a number of particles at most some maximal allowed number), the growth cluster is formed.

We present pictures, obtained by programming the models in Matlab, of growth clusters for finite n and $d = 2$. For all models, we obtain beautiful, self-similarly patterned shapes. The appeal of these pictures has been noted before, e.g., the sandpile pictures for $h = 0, -1$ can be found on the internet as "sandpile mandala", but the patterns are so far unexplained. Similar patterns are found in the so-called sandpile identity configuration [7, 33].

Finally, we explain our main limiting shape results for each model. For the sandpile model, we obtain that the limiting shape is a cube for $h = 2d-2$, (Theorem 5.4.1), and a sphere as $h \to -\infty$ (Theorem 5.4.8). This last result also applies for the double router model. We remark that the sandpile model with $h \to -\infty$, strongly resembles the divisible sandpile introduced by Levine and Peres [32], of which the limiting shape is also a sphere.

For the rotor router model, we find that the limiting shape is a sphere for all h (Theorem 5.4.3). Finally, we generalize the bounds of Le Borgne and Rossin for growth cluster of the sandpile model, to all h and d. As a corollary, we find that the rotor router shape contains a diamond, and is contained in a cube.

The rest of the paper contains proofs of these results. In Section 5.3, we derive inequalities for the growth clusters. In Section 5.4, we prove the various limiting shape theorems.

5.2 Model definitions and results

Before introducing each of the three models, we present some general definitions that apply to each model. All models are defined on $\mathbb{Z}^d$. We define a configuration $\eta = (H, T, D)$ as consisting of the following three components: the particle function $H : \mathbb{Z}^d \to \mathbb{Z}$, the toppling function $T : \mathbb{Z}^d \to \mathbb{Z}$ and the direction function $D : \mathbb{Z}^d \to \mathcal{D} = \{0, 1, \ldots, 2d-1\}$. We will use the $2d$ possible values in $\mathcal{D}$ to indicate the $2d$ unit vectors $\mathbf{e}_0 \ldots \mathbf{e}_{2d-1}$. The results will not depend on which value is assigned to which vector, as long as the assignment is fixed.

We call a configuration *allowed* if

1. For all $\mathbf{x}$, $T(\mathbf{x}) \geq 0$,
2. For all $\mathbf{x}$ with $T(\mathbf{x}) > 0$, $H(\mathbf{x}) \geq 0$,
3. For all $\mathbf{x}$ with $T(\mathbf{x}) = 0$, $H(\mathbf{x}) \geq h$,
4. For all $\mathbf{x}$, $D(\mathbf{x}) \in \mathcal{D}$.

Here, $h \in \mathbb{Z}$, the "background height", is a model parameter.

Each model starts with initial configuration $\eta_0 = \eta_0(n, h)$, which is as follows: $H_0(\mathbf{x}) = n$ if $\mathbf{x} = \mathbf{0}$, and $H_0(\mathbf{x}) = h$ otherwise, $T_0(\mathbf{x}) = 0$ for all $\mathbf{x}$, and $D_0(\mathbf{x}) \in \mathcal{D}$ for all $\mathbf{x}$. Observe that η_0 is allowed.

We now define a toppling as follows:

Definition 5.2.1. *A toppling of site* $\mathbf{x}$ *in configuration* η *consists of the following operations:*

- $T(\mathbf{x}) \to T(\mathbf{x}) + 1$,
- $H(\mathbf{x}) \to H(\mathbf{x}) - c$, *with* $c \leq 2d$ *some value specific for the model,*
- $H(\mathbf{x} + \mathbf{e}_i) \to H(\mathbf{x} + \mathbf{e}_i) + 1$, *with* $i = (D(\mathbf{x}) + 1) \bmod 2d, \ldots, (D(\mathbf{x}) + c) \bmod 2d$,
- $D(\mathbf{x}) \to (D(\mathbf{x}) + c) \bmod 2d$.

In words, in a toppling c particles from site $\mathbf{x}$ move to c different neighbors of $\mathbf{x}$, chosen according to the value of $D(\mathbf{x})$ in cyclic order. We call a toppling of site $\mathbf{x}$ *legal* if after the toppling $H(\mathbf{x}) \geq 0$. We call a site $\mathbf{x}$ stable if $H(\mathbf{x}) \leq h_{\max}$, with $h_{\max} = c - 1$, so that a site can legally topple only if it is unstable.

We can now define stabilization of a configuration as performing legal topplings, such that a stable configuration is reached, that is, a configuration where all sites are stable. We call this configuration the final configuration $\eta_n = \eta_n(n, h)$, which will of course depend on the model. To ensure that the final configuration is reached in a finite number of topplings, we impose for each model $h < h_{\max}$. For each model, it is then known that η_n does not depend on the order of the topplings. This is called the abelian property. The abelian property has been proved for centrally seeded growth models in general ([13], Section 4), and for the sandpile model in particular [39].

In the remainder of this paper, we will choose various orders of topplings. In Section 5.4.2, we will even admit illegal topplings. In Section 5.4.4, we will organize the topplings into discrete time steps, obtaining after time step t a configuration η^t, consisting of H^t, T^t and D^t.

We define two growth clusters that are formed during stabilization, denoting by $\mathbf{x}^\square$ the unit cube centered at $\mathbf{x}$ (i.e., $\mathbf{x}^\square = \{\mathbf{y} : \mathbf{y} = \mathbf{z} + \mathbf{x}, \mathbf{z} \in [-\frac{1}{2}, \frac{1}{2}]^d\}$):

Definition 5.2.2. *The toppling cluster is the cluster of all sites that have toppled, that is,*

$$\mathcal{T}_n = \bigcup_{\mathbf{x}: T_n(\mathbf{x}) > 0} \mathbf{x}^\square,$$

and the particle cluster is the cluster of all sites that have been visited by particles from the origin, that is,

$$\mathcal{V}_n = \mathcal{T}_n \ \cup \bigcup_{\mathbf{x}: H_n(\mathbf{x}) > h} \mathbf{x}^\square.$$

A cluster as a function of n has a limiting shape if, appropriately scaled, it tends to a certain shape as $n \to \infty$, in some sense to be specified later. By the model definition, all clusters are path connected. Observe that from Definitions 5.2.1 and 5.2.2, it follows that

$$\mathcal{T}_n \subset \mathcal{V}_n \subseteq \mathcal{T}_n \cup \partial\mathcal{T}_n, \tag{5.2.3}$$

with $\partial\mathcal{T}_n$ the exterior boundary of $\mathcal{T}_n$. We also define the "lattice ball":

$$\mathcal{B}_n = \bigcup_{i=1}^{n} \mathbf{x}_i^{\square}, \tag{5.2.4}$$

where the lattice sites $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of $\mathbb{Z}^d$ are ordered in such a way that the Euclidean distance from the origin is non-decreasing.

5.2.1 The rotor-router (RR,h) model

For the rotor router model, $c = 1$, so that in each toppling only one particle moves to a neighbor. Therefore, $h_{\max} = 0$, and the model is defined for $h < 0$ only. In words, every site holds the first $|h|$ particles that it receives, and sends every next particle to a neighbor, choosing the neighbors in a cyclic order. This means that after $2d$ topplings, every neighbor received a particle. Instead of a direction function with numerical values, we can think of an arrow being present at every site. In a toppling, the arrow is rotated to a new direction, and the particle is sent in this new direction.

Propp, Levine and Peres studied the case $h = -1$. Levine and Peres have proven that for $h = -1$, the limiting shape of the rotor router model is a Euclidean sphere. More precisely, they showed ([31], Thm. 1.1):

$$\lim_{n\to\infty} \lambda(n^{-1/d}\mathcal{V}_n \triangle B) = 0, \tag{5.2.5}$$

where λ denotes d-dimensional Lebesgue measure, $\triangle$ denotes symmetric difference, and B is the Euclidean sphere of unit volume centered at the origin in $\mathbb{R}^d$. This result is independent of D_0, and of any assignment of different unit vectors to possible values of $D(\mathbf{x})$. Note that the scaling function is necessarily $n^{-1/d}$, since $|\mathcal{V}_n| = n$ by model definition.

In [27], a picture of D_n is given for $h = -1$ and $n = 3$ million and D_0 constant. The picture shows a circular shape with an intriguing seemingly self-similar pattern. Curiously enough, this picture actually shows the shape of $\mathcal{T}_n$ rather than $\mathcal{V}_n$, indicating that the limiting shape of $\mathcal{T}_n$ is also a sphere. This would be a stronger statement than (5.2.5), but it remains as yet unproven.

It has been noted that the shape of $\mathcal{V}_n$ for the rotor-router model is remarkably circular, that is, as close to a circle as a lattice set can get, for every n. However, the shape of $\mathcal{T}_n$ has not been studied before. We programmed the model for several values of h and n, and observed for all these values that $\mathcal{V}_n \setminus \mathcal{T}_n$ is concentrated on the inner boundary of $\mathcal{V}_n$.

Our first main result of the rotor router model is the generalization of (5.2.5) to all $h \leq -1$, stated in Theorem 5.4.3. The proof in fact uses (5.2.5) as main ingredient.

By an entirely different method, in fact as a corollary of Theorem 5.4.11, we moreover obtain the result that the rotor router shape contains a diamond of radius proportional to $(\frac{n}{2d-1-h})^{1/d}$. It is surprising that, in spite of the limiting shape of the RR,h model being known, this is still a new result. But, in the words of Levine and Peres, the convergence in (5.2.5) does not preclude the formation of, e.g., holes close to the origin, as long as their volume is negligible compared to n. The only comparable previous result is that for $d = 2$ and $h = -1$, the particle cluster contains a disk of radius proportional to $n^{1/4}$ [30].

5.2.2 The abelian sandpile (SP,h) model

For the abelian sandpile model, $c = 2d$. Therefore, $h_{\max} = 2d - 1$, and the model is defined for $h < 2d - 1$. In each toppling, one particle moves to every neighbor, so that in fact the value of $D(\mathbf{x})$ is irrelevant.

It follows that for the SP model, we can specify (5.2.3) to

$$\mathcal{V}_n = \mathcal{T}_n \cup \partial\mathcal{T}_n. \tag{5.2.6}$$

The SP model as a growth model has received some attention in the cases $h = -1$ ("greedy" sandpile) and $h = 0$ ("non-greedy"), for which pictures of H_n can be found [27]. It is noted that the shape does not seem to be circular. In Figure 5.1, we show a family of sandpile pictures for a range of values of h, obtained by programming the model in Matlab. We see the number of symmetry axes increasing as h decreases. The shape appears to become more circular as h decreases. The shape for $h = 2$ is observed to be a square, but for the other values of h it seems to tend to a more complicated shape.

Again, we find that $\mathcal{V}_n \setminus \mathcal{T}_n$ is concentrated on the inner boundary of $\mathcal{V}_n$, for all observed values of h.

For the sandpile model, we have the following results. Theorem 5.4.1 states that indeed the toppling cluster for the SP,$2d-2$ model is a (d-dimensional) cube, and the particle cluster tends to a cube as $n \to \infty$.

Theorem 5.4.8 states that for $h \to -\infty$, the limiting shape is a sphere.

Finally, we generalize some bounds for the scaling function that have been obtained by Le Borgne and Rossin [7] in the case $d = 2$, $0 \leq h \leq 2$, to all d and h. This result is formulated in Theorem 5.4.11. Our proof moreover allows to deduce that $\mathcal{V}_n$ for the sandpile model is simply connected for all n.

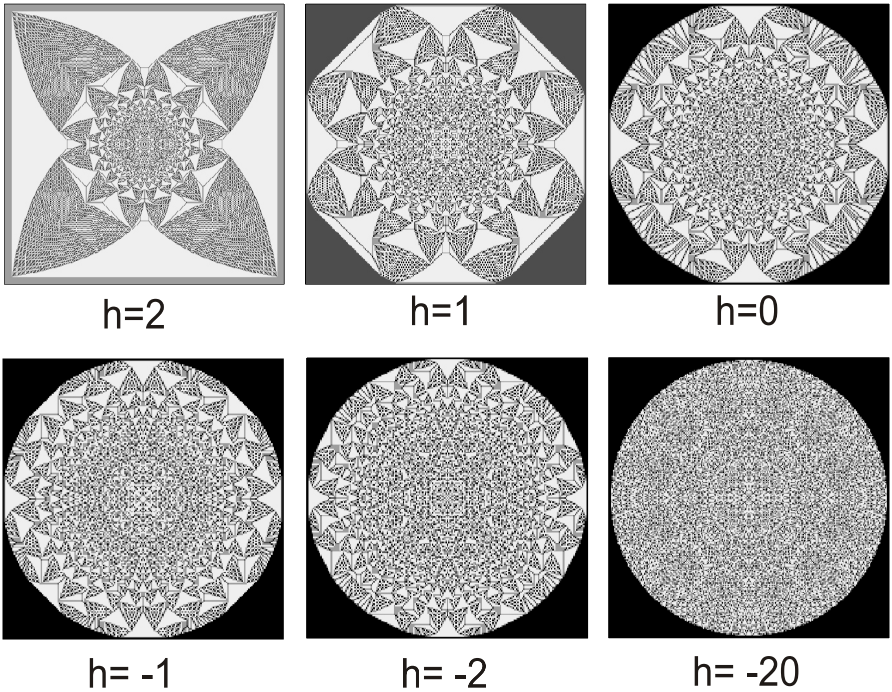

Figure 5.1: H_n for the sandpile model with different h values. The number of particles ranges from $n = 60{,}000$ ($h = 2$) to 1,000,000 ($h = -20$). The gray-scale colors are such that a lighter color corresponds to a higher value of $H(\mathbf{x})$.

5.2.3 The double router (DR,h) model

In the double router model, $c = 2$. Therefore, $h_{\max} = 1$, and the model is defined for $h \leq 0$. In a toppling of this model, two particles are sent out in two different directions, such that after d topplings, every neighbor received a particle.

Many variants of this model are possible, e.g., for $d = 3$ one could choose $c = 3$, which amounts to dividing the 6 neighbors in 2 groups of 3. However, to avoid confusion, we only discuss the above explained variant.

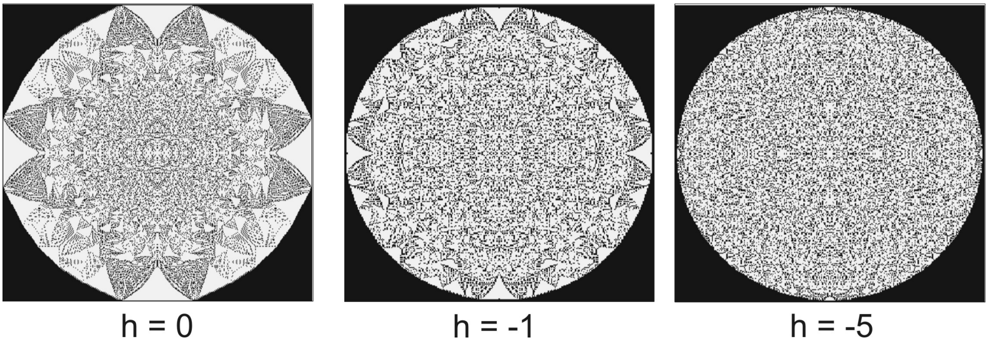

Figure 5.2: H_n for the DR model with $h = 0$ ($n = 110{,}000$), $h = -1$ ($n = 100{,}000$), and $h = -5$ ($n = 400{,}000$), and $D_0(\mathbf{x}) = 0$ for all $\mathbf{x}$. Sites with height 1 are colored white, sites with height 0 or negative are colored black.

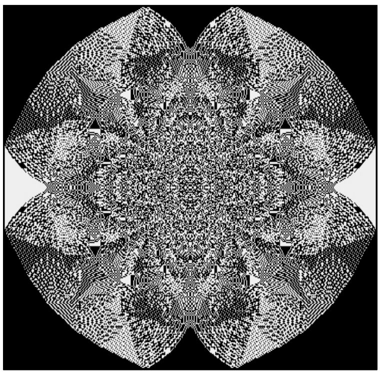

Figure 5.3: D_n for the DR model with $h = 0$ ($n = 110{,}000$), and $D_0(\mathbf{x}) = 0$ for all $\mathbf{x}$. White corresponds to $D(\mathbf{x}) = 2$, black to $D(\mathbf{x}) = 0$.

Figure 5.2 shows H_n for $h = 0$, $d = 2$ and $n = 110.000$. We ordered the unit vectors as $\mathbf{e}_0$ = left, $\mathbf{e}_1$ = right, $\mathbf{e}_2$ = up, $\mathbf{e}_3$ = down. Initially, $D(\mathbf{x}) = 0$ for all $\mathbf{x}$. Figure 5.3 shows D_n for the same case, to indicate that for this model we find complex patterns both for H_n and D_n.

As for the sandpile model, we see a symmetric, yet not circular shape, with notable flat edges. In Figure 5.2, also some other values of h are shown, again indicating that the shape seems to become more and more circular with decreasing h. Indeed, Theorem 5.4.8 states this fact for both the sandpile model and the double router model.

Again, we observe that $\mathcal{V}_n \setminus \mathcal{T}_n$ is concentrated on the inner boundary of $\mathcal{V}_n$, for all observed values of h.

5.3 Comparing the models

From the model descriptions, it is clear that one SP toppling always equals d DR topplings, as well as $2d$ RR topplings. Furthermore, our toppling definition (Definition 5.2.1) ensures that one DR toppling always equals two consecutive RR topplings. We exploit these relations to compare the clusters for different models. We will with a subindex RR, DR or SP indicate the model that was used to obtain the growth cluster, and also use a subindex h because we compare different h values.

Proposition 5.3.1. *For every fixed h (for which the model is defined), d and n,*

1. $\mathcal{V}_{n,SP,h} \subseteq \mathcal{V}_{n,DR,h} \subseteq \mathcal{V}_{n,RR,h}$,
2. *For all models i = RR, DR and SP:* $\mathcal{V}_{n,i,h-1} \subseteq \mathcal{V}_{n,i,h}$,
3. $\mathcal{V}_{n,RR,h-1} \subseteq \mathcal{V}_{n,DR,h}$, *and* $\mathcal{V}_{n,RR,h-(2d-1)} \subseteq \mathcal{V}_{n,SP,h}$.

Proof. The proof makes use of the abelian property of all models, that is, the property that the stabilized configuration η_n does not depend on the order of topplings. We are therefore free to choose a convenient order. Furthermore, a site $\mathbf{x}$ that at some instant during stabilization has either $T(\mathbf{x}) > 0$, or $T(\mathbf{x}) = 0$ and $H(\mathbf{x}) > h$, must belong to $\mathcal{V}_n$ in the final configuration, because by further topplings either $H(\mathbf{x})$ or $T(\mathbf{x})$, or both, increase.

part 1. The initial configurations for all these three models are the same. We first compare $\mathcal{V}_{n,SP,h}$ with $\mathcal{V}_{n,DR,h}$. We choose, for both the SP and the DR model, to first perform all legal SP-topplings. Since every SP-toppling consists of d DR-topplings, these are legal topplings for both models. The configuration is now stabilized for the SP model, but there can be sites in $\mathcal{V}_{n,SP,h}$ that are not stable in the DR model, since h_{max} is $2d - 1$ for the SP model and 1 for the

DR model. Therefore, in the MD model, possibly more topplings follow, so that $\mathcal{V}_{n,SP,h} \subseteq \mathcal{V}_{n,DR,h}$.

We compare $\mathcal{V}_{n,DR,h}$ with $\mathcal{V}_{n,RR,h}$ in an analogous manner, this time using that h_{max} is 1 for the DR model and 0 for the RR,h model, to get $\mathcal{V}_{n,DR,h} \subseteq \mathcal{V}_{n,RR,h}$.

part 2. We start with comparing the RR,$h-1$ model with the RR,h model. We choose, for both initial configurations, to first perform all RR-topplings that stabilize the initial configuration of the RR,$h-1$ model. These are legal topplings for both configurations. The configuration is now stable for the RR,$h-1$ model, but all sites that have $H(\mathbf{x}) = 0$ for this model, have $H(\mathbf{x}) = 1$ for the RR,h model. Therefore, for this model more topplings would follow, so that $\mathcal{V}_{n,RR,h-1} \subseteq \mathcal{V}_{n,RR,h}$. The same reasoning can be applied for the DR and SP model.

part 3. We first compare $\mathcal{V}_{n,DR,h}$ with $\mathcal{V}_{n,RR,h-1}$. We choose, for both models, to first perform all DR-topplings that would stabilize the initial configuration with height $h-1$. These are legal topplings for both models. Then for both models, more topplings are needed. For the RR,$h-1$ model there can be sites with $H(\mathbf{x}) = 1$, that are unstable. For the DR,h model, the same sites have $H(\mathbf{x}) = 2$, and therefore are also unstable. But since one DR-toppling equals two RR-topplings, the set of sites where the configuration changes by the extra topplings for the RR,$h-1$ model, is a subset of those for the DR,h model. Therefore, $\mathcal{V}_{n,RR,h-1} \subseteq \mathcal{V}_{n,DR,h}$. The argument for the sandpile model is similar. $\square$

5.4 Limiting shape results

5.4.1 The sandpile model in $\mathbb{Z}^d$, with $h = 2d-2$

As noted in Section 5.2, Figure 5.1 indicates that the limiting shape for the sandpile model with $h = 2$ and $d = 2$ is a square. This section contains the proof of a more general statement for arbitrary dimension, that is, we prove that indeed the toppling cluster for the SP,$2d-2$ model is a (d-dimensional) cube, and the particle cluster tends to a cube as $n \to \infty$. However, we have no explicit expression for the scaling function $f(n)$, so that the scaled clusters $f(n)\mathcal{V}_n$ and $f(n)\mathcal{T}_n$ would tend to the unit cube, just that this scaling satisfies $n^{-1} \leq f(n) \leq \frac{1}{2}n^{-1/d} - \frac{3}{2}$. Based on the calculations, we believe that $f(n)$ is $O(n^{-1/d})$.

Theorem 5.4.1. *Let $\mathfrak{C}(r)$ be the cube $\bigcup_{\mathbf{x}:\max_i |x_i| \leq r} \mathbf{x}^{\square}$. For every n in the d-dimensional SP,$2d-2$ model, there is an r_n such that*

$$\mathcal{T}_n = \mathfrak{C}(r_n),$$

and

$$\mathcal{V}_n = \mathfrak{C}(r_n) \cup \partial\mathfrak{C}(r_n).$$

For all n, this r_n satisfies

$$\frac{1}{2}n^{1/d} - \frac{3}{2} \leq r_n \leq n.$$

We start by outlining the case $d = 2$ as an example, for the sake of clarity. When simulating the model for several small values of n - this can even be done by hand - one notices that the configuration is always such that it contains a central square with all boundary sites full, except for the corner sites, which have height $h_{max} - 1$. Outside this square, all sites have height $h_{max} - 1$. We will call such a rectangular boundary a critical boundary. This is all the information we need to make an inductive argument in n. Suppose, η_n has a critical boundary, we add one grain to the origin and in the course of stabilization (to obtain η_{n+1}), one boundary site topples. As a consequence, all neighboring boundary sites topple, because they are all full and all in turn receive a grain. The two adjacent corner sites receive one grain, therefore become full. Therefore, after these topplings a new rectangular critical boundary is created. If any more boundary sites topple, we can reiterate this argument. We conclude that the presence of a rectangular critical boundary is stable under additions inside this rectangle. In the special case where additions are made only to the origin, we conclude by symmetry that the shape of the critical boundary will always be square.

Below, we schematically show the argument for $n = 7$.

2	2	2	2	2
2	2	3	2	2
2	3	4	3	2
2	2	3	2	2
2	2	2	2	2

$\longrightarrow$

2	2	2	2	2
2	2	4	2	2
2	4	0	4	2
2	2	4	2	2
2	2	2	2	2

$\longrightarrow$

2	2	3	2	2
2	4	0	4	2
3	0	4	0	3
2	4	0	4	2
2	2	3	2	2

$\longrightarrow$

We start with η_7 plus one extra grain at the origin. Full boundary sites are colored lightgrey, unstable sites grey. First the origin topples, causing the full boundary sites to become unstable. Next, these boundary sites topple, causing the corner sites to be unstable. The first full sites of the new boundary are created.

2	3	3	3	2
3	0	2	0	3
3	2	4	2	3
3	0	2	0	3
2	3	3	3	2

$\longrightarrow$

2	3	3	3	2
3	0	3	0	3
3	3	0	3	3
3	0	3	0	3
2	3	3	3	2

Next, the corner sites topple. The configuration now has a new square critical boundary. A further toppling of the origin, after which we obtain η_8, does not change this new boundary anymore.

In the proof below, we generalize this critical boundary to arbitrary d and make the details more precise, but the idea will remain the same.

In the course of the proof, we will need the following lemma.

Lemma 5.4.2. *Let ϕ_r be the following configuration: $H(\mathbf{0}) = 2d$, and for all other $\mathbf{x} \in \mathfrak{C}_r$, $H(\mathbf{x}) = h_{max}$, otherwise $H(\mathbf{x}) = h_{max} - 1$. If the configuration ϕ_r is stabilized, then during stabilization, every site $\mathbf{x}$ topples exactly $\delta_{\mathbf{x},r} = \max\{r + 1 - \max_i |x_i|, 0\}$ times.*

Proof. We choose to order the topplings into *waves* [24], that is, in each wave, we topple the origin once and then all other sites that become unstable, except the origin again. No site can topple more than once during a wave.

In the first wave, all sites in $\mathfrak{C}_r$ topple once, because it is maximally filled. The wave stops at the boundary, because the sites outside $\mathfrak{C}_r$ have at most one toppling neighbor, therefore they can not become unstable. All sites in $\mathfrak{C}_{r-1}$ then have $2d$ once-toppling neighbors, so their particle number does not change. The sites in $\mathfrak{C}_r \setminus \mathfrak{C}_{r-1}$ have less toppling neighbors, so their particle number becomes at most $2d - 2$.

Therefore, the effect of the first wave on ϕ_r, is to make it, for all sites in $\mathfrak{C}_r$, at most equal to ϕ_{r-1}, with equality for all sites with $\mathfrak{C}_{r-1}$. It follows that the next wave will topple all sites in $\mathfrak{C}_{r-1}$ once. Continuing this argument, the result stated in the lemma follows. $\square$

Proof of Theorem 5.4.1. We will use induction in n. For that, we choose to obtain the final configuration as follows: the n particles are added to the origin one by one, each time first stabilizing the current configuration through topplings. Due to abelianness, this procedure will give the same final configuration as when all n particles are added simultaneously, before toppling starts. We will show that during this procedure, only configurations in $\mathcal{C}_r$ are encountered, where $\mathcal{C}_r$ is the set of configurations that are as follows:

1. $\mathcal{T}_n = \mathfrak{C}_r$,

2. For all $\mathbf{x} \in \mathcal{T}_n$ such that $\max_i |x_i| = r$, $T_n(\mathbf{x}) = 1$,

3. $\mathcal{V}_n = \mathfrak{C}_r \cup \partial\mathfrak{C}_r$,

4. For all $\mathbf{x} \in \mathcal{V}_n \setminus \mathcal{T}_n$, $H_n(\mathbf{x}) = h_{max} = 2d - 1$,

5. For all $\mathbf{x} \notin \mathcal{V}_n$, $H_n(\mathbf{x}) = 2d - 2$.

In words, a configuration in $\mathcal{C}_r$ is such that $\mathcal{T}_n$ is a cube, where all inner boundary sites toppled exactly once, and all outer boundary sites, i.e., sites that do not belong to $\mathcal{T}_n$ but have a neighbor in $\mathcal{T}_n$, have particle number h_{max}. Thus, $\mathcal{T}_n$ is surrounded by $2d$ maximally filled square "slabs" of size $(2r+1)^{d-1}$. All sites outside $\mathcal{V}_n$ have particle number $2d-2$, by model definition.

After the first particle is added, $H_1(\mathbf{0}) = h_{max}$, and no site has toppled yet. After the second particle is added, the origin topples once, and all neighbors of the origin receive one particle. Therefore, $\eta_2 \in \mathcal{C}_0$.

We will now show that if we assume $\eta_n \in C_r$ for some r, then either $\eta_{n+1} \in C_r$ or $\eta_{n+1} \in C_{r+1}$. To prove this, we will add a particle at the origin to configuration η_n and start stabilizing. First we show that if during stabilization the configuration leaves $\mathcal{C}_r$, then it enters $\mathcal{C}_{r+1}$. Then we show that in this last case, it does not leave $\mathcal{C}_{r+1}$ during further stabilization.

If no site $\mathbf{x} \in \mathcal{V}_n$ such that $\max_i |x_i| = r$ topples during stabilization, then η_{n+1} remains in C_r. But if one such site topples, then one site in $\mathcal{V}_n \setminus \mathcal{T}_n$ becomes unstable, and also topples. This site is in one of the $2d$ maximally filled slabs. If one site of such a slab topples, then the entire slab must topple, because all of its sites will in turn receive a particle from a toppling neighbor. After the entire slab toppled once, all neighbors of the slab received a particle, and the configuration can be described as follows: The toppled cluster is a rectangle of size $(2r+1)^{d-1}(2r+2)$, surrounded by maximally filled slabs, two of them cubic and the rest rectangular.

But due to symmetry, if one slab topples once then this must happen for all slabs. We can choose in what order to topple the slabs; suppose we divide them in d opposite pairs. After we toppled the first pair of slabs, the configuration is a central rectangle of size $(2r+1)^{d-1}(2r+3)$, surrounded by maximally filled slabs, two of them cubic and the rest rectangular. After we toppled the k^{th} pair of (meanwhile possibly rectangular) slabs, the configuration is a central rectangle of size $(2r+1)^{d-k}(2r+3)^k$, surrounded by maximally filled slabs, so that after all slabs toppled, the toppled cluster is again a cube, now of size $(2r+3)^d$, centered at the origin and surrounded by $2d$ maximally filled square slabs. In other words, after these topplings the configuration is in $\mathcal{C}_{r+1}$. It follows that if $\eta_n \in \mathcal{C}_r$ for some r, then $\eta_{n+1} \in \mathcal{C}_{r'}$, for some $r' \geq r$. It now remains to show that r' can only have the values r or $r+1$.

We use lemma 5.4.2. A configuration $\eta \in \mathcal{C}_r$, plus an addition at the origin, has at every site at most the number of particles given by ϕ_{r+1}. Therefore, if we suppose $\eta_n \in \mathcal{C}_r$, then upon addition of a particle at the origin, by abelianness, the number of topplings for every $\mathbf{x}$ will at most be $\delta_{\mathbf{x},r+1}$. In particular, the outer boundary sites of $\mathcal{T}_n$ will topple at most once.

The induction proof is now completed, therefore we now know that, for all n, $\eta_n \in \mathcal{C}_r$ for some r. We have also shown that $r_n \leq n$. From the description of $\mathcal{C}_r$, we see that $\mathcal{T}_n$ is always a cube, and that $\mathcal{V}_n$ is more and more like a cube as r_n increases.

To conclude that $\mathcal{V}_n$ tends to a cube as $n \to \infty$, we finally need to show that r_n increases with n. For this, note that in η_n every site needs to be stable. Since $h = h_{max} - 1$, every site can accommodate at most one extra particle. Therefore, if $\eta_n \in C_r$, then $r \geq \frac{1}{2}n^{1/d} - \frac{3}{2}$. □

5.4.2 The rotor-router model with $h < -1$

In this section, we will prove the following result:

Theorem 5.4.3. *The limiting shape of the particle cluster for the rotor-router model is a sphere for every $h \leq -1$. More precisely,*

$$\lim_{n\to\infty} \lambda\left((\frac{n}{|h|})^{-1/d}\mathcal{V}_n \triangle B \right) = 0.$$

The proof will also reveal the following about the rotor-router toppling cluster:

Corollary 5.4.4. *The toppling cluster of the RR,h model contains a cluster $\mathcal{W}_n$, with*

$$\lim_{n\to\infty} \lambda\left((\frac{n}{|h|+1})^{-1/d}\mathcal{W}_n \triangle B \right) = 0.$$

Theorem 5.4.3 has been proven for the case $h = -1$, by Levine and Peres, see (5.2.5). We will use their result to prove the theorem for other values of h. The strategy of the proof will be as follows. First, we introduce a slightly different version of the RR,h model, which we will call the k-color rotor router, or RR',k model. We show that the limiting shape of this model is a sphere for all k, with scaling function $(\frac{n}{k})^{-1/d}$. Informally, the k-color model can be viewed as k iterations of the RR,-1 model. The n particles at the origin are equally divided in k different-colored groups, and for each color the particles spread out until there is at most one particle of each color at each site. From (5.2.5), we then know that each color region is approximately spherical, so we get a final configuration that looks like k almost completely overlapping, approximately spherical, different-colored regions.

Then we show that $\mathcal{V}_{n,RR,h}$ and $\mathcal{V}_{n,RR',k}$, with $h = -k$, differ only in a number of sites that is $o(n)$. This number is at most the number of sites in $\mathcal{V}_{n,RR',k}$ where not every color is present. In the proof of this second point, we will stabilize η_0 for the RR,h model by first performing all topplings needed to stabilize the RR',$-h$ model. Some of these topplings may be illegal for the RR,h model, so we will reverse them by performing untopplings. The main difficulty in the proof is to show that we do reach η_n by this procedure. First we define untopplings.

Definition 5.4.5. *An untoppling of site* $\mathbf{x}$ *in configuration* η *consists of the following operations:*

- $T(\mathbf{x}) \to T(\mathbf{x}) - 1$,
- $H(\mathbf{x}) \to H(\mathbf{x}) + c$, *with* c *according to the model,*
- $D(\mathbf{x}) \to (D(\mathbf{x}) - c) \bmod 2d$,
- $H(\mathbf{x} + \mathbf{e}_i) \to H(\mathbf{x} + \mathbf{e}_i) - 1$, *with* $i = D(\mathbf{x}), (D(\mathbf{x}) + 1) \bmod 2d, \ldots, (D(\mathbf{x}) + c - 1) \bmod 2d$.

We call an untoppling of site $\mathbf{x}$ *legal* if $T(\mathbf{x}) > 0$ before the untoppling. We see that a legal untoppling of site $\mathbf{x}$ is precisely the undoing of a toppling of site $\mathbf{x}$, in the sense that the particles that were sent out of site $\mathbf{x}$ in the last toppling, return to this site, and the value of $D(\mathbf{x})$ returns to its previous value.

The following proposition provides a more elaborate description of η_n in the case where we allow illegal topplings, that we will need in the proof of Theorem 5.4.3 (we remark that the abelian property still holds for combinations of legal and illegal topplings). We introduce the notion of optimality to distinguish η_n from configurations that are stable and allowed, but that cannot be obtained from η_0 by legal topplings. An example for the rotor router model is as follows. Suppose that, after reaching η_n by legal topplings, there is somewhere a closed loop of sites with height 0 such that "the arrows form a cycle", i.e., if we topple all of them, they would all send and receive one particle. After these topplings (of which at least the first would be illegal), the height function is still H_n, but the toppling number of these sites has increased by 1, so we obtained a different, stable and allowed configuration. We remark that a set of sites forming such a "cycle of arrows" cannot be found in $\mathcal{T}_n$. Since all sites that toppled are full, in the last toppling in this set a particle must have left the set, so that there is an arrow pointing out of the set (see also [43], section III). Therefore, if there is a set of sites forming a cycle of arrows, then at least one of these sites did not topple.

Proposition 5.4.6. *Call a stable configuration* optimal *if every sequence of legal untopplings leads to an unstable configuration. For every model,* η_n *is the unique optimal, stable and allowed configuration that can be reached from* η_0 *by topplings, either legal or illegal, and legal untopplings.*

Proof. We defined η_n before as the unique stable configuration that can be reached from η_0 by legal topplings. It follows from this definition that η_n is allowed, since if sites can only topple when they are unstable, then we automatically obtain for all $\mathbf{x}$ where $T_n(\mathbf{x}) > 0$, that $H_n(\mathbf{x}) \geq 0$.

To prove that η_n is optimal, we proceed by contradiction. Suppose that, starting from η_n, there is a sequence of legal untopplings such that a stable configuration ξ is obtained. Then ξ can be obtained from η_0 by a sequence of legal or illegal topplings. Call T' the toppling function for this sequence, then $0 \leq T'(\mathbf{x}) \leq T_n(\mathbf{x})$ for all $\mathbf{x}$ (because we undid some topplings of T_n), but $T' \neq T_n$. By abelianness, ξ depends only on η_0 and T'. Thus, we can choose the order of the topplings according to T', to obtain ξ. There cannot be an order such that all topplings are legal, otherwise ξ cannot be different from η_n. We will therefore choose the order such that first all possible legal topplings are performed, and then the rest (as an example, suppose $T'(\mathbf{0}) = 0$. Then we must start with an illegal toppling, since in η_0 all sites but the origin are stable). After all possible legal topplings according to T', we have a configuration with at least one unstable site, since we did not yet reach η_n. In the remainder, this site is not toppled, because otherwise another legal toppling could have been added to the first legal topplings. Therefore, ξ cannot be stable.

We so far established that η_n is optimal, stable and allowed. Now we prove that it is unique by deriving another contradiction. Suppose there is an other optimal, stable and allowed configuration ζ that can be reached from η_0 by topplings and legal untopplings. Call T'' the toppling function for this sequence. Then $T''(\mathbf{x}) \geq T_n(\mathbf{x})$ for all $\mathbf{x}$. We can see this as follows: we choose the toppling order such that we first perform all topplings that are also in T_n. If there were some sites $\mathbf{y}$ where $T''(\mathbf{y}) < T_n(\mathbf{y})$, then at this point at least one of these sites would be unstable. But in the remainder of T'', this site would not topple again, so in that case ζ could not be stable.

Call $\tau(\mathbf{x}) = T''(\mathbf{x}) - T_n(\mathbf{x})$. Since $\zeta \neq \eta_n$, $T'' \neq T_n$. Suppose we first perform the topplings according to T_n, and then according to τ. Then we have that ζ can be reached from η_n by performing $\tau(\mathbf{x})$ topplings for every site $\mathbf{x}$, and vice versa, that η_n can be reached from ζ by performing $\tau(\mathbf{x})$ legal untopplings for every site $\mathbf{x}$. But then ζ cannot be optimal. □

Proof of Theorem 5.4.3. The RR',k model is defined as follows: As in the RR,h model, $c = 1$. Initially, there are n particles at the origin; at every other site there are $-k$ particles. We choose n a multiple of k, and divide the n particles into k groups of n/k particles, each group with a different color. The height $H(\mathbf{x})$ of a site $\mathbf{x}$ is now defined as $-k$ plus the total number of particles present at $\mathbf{x}$, of either color. However, now we call a site stable if it contains at most one particle of each color. If a particle arrives at a site where its color is already present, then that particle will be sent to a neighbor in the subsequent toppling. Note that in this model, a site $\mathbf{x}$ can be unstable even if $H(\mathbf{x}) < 0$. We furthermore restrict the order of legal topplings for this model. We say that in this model, first all legal topplings for the first color should be performed, then for the next, etc. Then for each color, the model behaves just as the $RR, -1$ model, with for each color a new initial T and D. Therefore, with this toppling order the final configuration consisting of H_n, T_n and D_n, is well-defined.

We now show that the limiting shape of the RR',k model is a sphere. We perform first all the topplings with particles of the first color, say, red. From [31], Thm. 2.1 it follows that these particles will form a cluster $\mathcal{V}^1_{n/k}$ close to the lattice ball $\mathcal{B}_{n/k}$. In fact, the number of sites in $\mathcal{V}^1_{n/k} \triangle \mathcal{B}_{n/k}$ is $o(n)$. We will denote this as

$$|\mathcal{V}^1_{n/k} \triangle \mathcal{B}_{n/k}| \leq f(n/k) = o(n).$$

D is now different from D_0. However, if next we stabilize for the blue particles, these will again form a cluster close to the lattice ball $\mathcal{B}_{n/k}$, since the result (5.2.5) does not depend on the initial D. Of course, this cluster $\mathcal{V}^2_{n/k}$ need not be the same as $\mathcal{V}^1_{n/k}$.

When all sites are stable, we have $\mathcal{V}_{n,RR',k} = \bigcup_{i=1}^k \mathcal{V}^i_{n/k}$. We also define a cluster $\mathcal{W}_{n,k} = \bigcap_{i=1}^k \mathcal{V}^i_{n/k}$. From the above, the number of sites in the difference of both these clusters with $\mathcal{B}_{n/k}$, is at most $kf(n/k)$. Thus

$$|\mathcal{V}_{n,RR',k} \triangle \mathcal{B}_{n/k}| \leq kf(n/k) = o(n). \tag{5.4.7}$$

We will call $\mathcal{X}_n = \mathcal{V}_{n,RR',k} \setminus \mathcal{W}_{n,k}$, so that $\mathcal{X}_n$ contains $|\mathcal{X}_n| \leq 2kf(n/k)$ sites. Sites in $\mathcal{W}_{n,k}$ contain k particles, sites in $\mathcal{X}_n$ contain less than k particles.

Now we compare this model with the RR,h model, with $h = -k$. Disregarding the colors, the initial configuration is the same for both models. Suppose we perform in the RR,h model all the topplings that are performed as above in the RR',$-h$ model. The configuration is then as follows: for all $\mathbf{x} \in \mathcal{W}_{n,k}$, $H(\mathbf{x}) = 0$ and $T(\mathbf{x}) \geq 0$, and for all $\mathbf{x}' \in \mathcal{X}_n$, $-k \leq H(\mathbf{x}') < 0$ and $T(\mathbf{x}') \geq 0$.

This configuration is possibly not η_n for the RR,h model, since it is possibly not allowed, if there are sites $\mathbf{x}'$ in $\mathcal{X}_n$ with $T(\mathbf{x}') > 0$. We can reach an allowed configuration by performing legal untopplings; it will appear that we then in fact reach η_n.

Suppose first that only one untoppling is needed. It might be that the neighbor $\mathbf{y}$ that returned a particle, now has less than k particles, so that $H(\mathbf{y}) < 0$, while $T(\mathbf{y}) > 0$. In that case, $\mathbf{y}$ would now also have to untopple. We call this process an untoppling avalanche. The avalanche stops if an allowed (stable) configuration is encountered, that is, if the last particle came from a site that did not topple. An untoppling avalanche consists of untoppling neighbors, each one passing a particle to the previous one. Therefore, an untoppling avalanche of arbitrary length, changes the particle number of only two sites in the configuration.

In our case, at most $(k-1)|\mathcal{X}_n|$ untoppling avalanches are required. Each avalanche stops if the last particle came from a site that did not topple. This will change the configuration in at most $2(k-1)|\mathcal{X}_n|$ sites. After all these untopplings, the configuration is allowed and stable, i.e., we have for all sites with $T(\mathbf{x}) > 0$, that $H(\mathbf{x}) = 0$. We can only perform legal untopplings at sites with $T(\mathbf{x}) > 0$, so that we cannot perform a sequence of legal untopplings in a closed loop of sites. Therefore, any further sequence of legal untopplings would increase $H(\mathbf{x})$ for at least one of these sites, and thus lead to an unstable configuration.

We have thus reached an optimal configuration, so we have reached η_n. Therefore, at this point we can conclude

$$|\mathcal{V}_{n,RR,h} \triangle \mathcal{V}_{n,RR',k}| \leq 2(k-1)|\mathcal{X}_n| \leq 4k(k-1)f(n/k) = o(n).$$

Combined with (5.4.7), this leads to Theorem 5.4.3. □

To prove the corollary, we define $\mathcal{W}_{n,h} = \bigcup_{\mathbf{x}:\mathbf{x}\in\mathcal{V}_{n,RR,h},H_n(x)=|h|} \mathbf{x}^{\square}$. By the above argument, we have, as for $\mathcal{W}_{n,k}$, that $|\mathcal{W}_{n,h} \triangle \mathcal{B}_{n/|h|}| = o(n)$.

Suppose, in the RR,h model, we first perform all topplings that are needed to stabilize for the RR,$h-1$ model, as in the proof of Proposition 5.3.1 part 2. The particle cluster then contains a cluster $\mathcal{W}_{n,h-1}$, with $\lim_{n\to\infty} \lambda\left((\frac{n}{|h-1|})^{-1/d}\mathcal{W}_n \triangle B\right) = 0$.

Because the sites in $\mathcal{W}_{n,h-1}$ contain $|h-1| = |h|+1$ particles, every site in $\mathcal{W}_{n,h-1}$ is unstable in the RR,h model. Therefore, $\mathcal{W}_{n,h-1} \subseteq \mathcal{T}_{n,RR,h}$. □

5.4.3 The double router model, and the sandpile model with $h \to -\infty$

In discussing Figures 5.1 and 5.2, we observed that the DR and SP shapes seem to become more circular as h decreases. Proposition 5.3.1 and Theorem 5.4.3 can indeed be combined to give the following result:

Theorem 5.4.8. *The limiting shape of the SP,h and the DR,h model, for $h \to -\infty$, is a sphere. More precisely,*

$$\lim_{h\to-\infty} \limsup_{n\to\infty} \lambda\left((\frac{n}{|h|})^{-1/d}\mathcal{V}_n \triangle B\right) = 0.$$

Proof. From Proposition 5.3.1, for all $h \leq -1$,

$$\mathcal{V}_{n,RR,h-(2d-1)} \subseteq \mathcal{V}_{n,SP,h} \subseteq \mathcal{V}_{n,RR,h}, \tag{5.4.9}$$

and

$$\mathcal{V}_{n,RR,h-1} \subseteq \mathcal{V}_{n,DR,h} \subseteq \mathcal{V}_{n,RR,h}. \tag{5.4.10}$$

We will discuss the DR,h model first.

From Theorem 5.4.3, we know that the particle cluster of the RR,h model, scaled by $(\frac{n}{|h|})^{-1/d}$, tends for every fixed value of h to the unit volume sphere as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \lambda \left((\frac{n}{|h|})^{-1/d} \mathcal{V}_{n,RR,h} \triangle B \right) = 0,$$

and thus we also know (note that since h is negative, $|h-1| = |h|+1$)

$$\lim_{n\to\infty} \lambda \left((\frac{n}{|h|})^{-1/d} \mathcal{V}_{n,RR,h-1} \triangle B \right) = \lambda \left(\left[(\frac{|h|}{|h|+1})^{1/d} B \right] \triangle B \right) = 1 - \frac{|h|}{|h|+1}.$$

Because of (5.4.10), it follows that

$$0 \le \limsup_{n\to\infty} \lambda \left((\frac{n}{|h|})^{-1/d} \mathcal{V}_{n,DR,h} \triangle B \right) \le 1 - \frac{|h|}{|h|+1},$$

so that

$$\lim_{h\to-\infty} \limsup_{n\to\infty} \lambda \left((\frac{n}{|h|})^{-1/d} \mathcal{V}_{n,DR,h} \triangle B \right) = 0.$$

The argument is similar for the SP,h model. □

5.4.4 The toppling cluster for the sandpile model

In this subsection, we generalize Theorem 4 of Le Borgne and Rossin [7], who studied the particle cluster of the SP,h model for $d = 2$ and $h = 0, 1$ and 2, to arbitrary d and h.

Theorem 5.4.11. *Let $\mathfrak{D}(r)$ be the diamond $\bigcup_{\mathbf{x}:\sum_{i=1}^d |x_i| \le r} \mathbf{x}^\square$, and $\mathfrak{C}(r)$ the cube $\bigcup_{\mathbf{x}:\max_i |x_i| \le r} \mathbf{x}^\square$.*

1. *In the SP,h model, for every n and $h \le 2d-2$, there is an r_n such that*

$$\mathfrak{D}(r_n - 1) \subseteq \mathcal{T}_n \subseteq \mathfrak{C}(r_n).$$

and

$$\mathfrak{D}(r_n) \subseteq \mathcal{V}_n \subseteq \mathfrak{C}(r_n + 1).$$

2. *This r_n satisfies*

$$\begin{aligned} \left(\tfrac{n}{2d-1-h}\right)^{1/d} - 3 \le 2r_n \qquad & \textit{for all } n \textit{ and } h \le 2d-2, \\ 2r_n \le \left(\tfrac{dn}{d-h}\right)^{1/d} + o(n^{1/d}) \quad & \textit{for } n \to \infty \textit{ and } h < d. \end{aligned}$$

Corollary 5.4.12. *$\mathcal{V}_{n,RR,h}$ contains the diamond $\mathfrak{D}\left(\frac{1}{2}\left(\frac{n}{2d-1-h}\right)^{1/d}-\frac{3}{2}\right)$ for all $h<0$, and is contained in the cube $\mathfrak{C}(\frac{1}{2}(\frac{dn}{d-h})^{1/d})$ for all $h<-d$.*

The corollary follows from combining the theorem with Proposition 5.3.1 part 1 and 3.

Remark 5.4.13. Note that the first inequality for r_n agrees with that in Theorem 5.4.1, for the sandpile model with $h = 2d-2$. We can not use the second inequality in this case, because that is only valid for $h < d$. From combining Theorem 5.4.1 with Proposition 5.3.1 part 3, we find that for $-d \leq h < 0$, $\mathcal{V}_{n,RR,h}$ is contained in the cube $\mathfrak{C}(n+1)$.

Proof of Theorem 5.4.11, part (1). We prove the inequality for $\mathcal{T}_n$. The inequality for $\mathcal{V}_n$ then follows from (5.2.6).

We follow the method of Le Borgne and Rossin, who introduce the following stabilization procedure: Starting at $t=0$ in the initial configuration with n particles at the origin, each time step every unstable site of the current configuration is stabilized. For example, to obtain η^1 we topple only the origin, as many times as legally possible. At $t=2$, we topple only the neighbors of the origin, et cetera. We will call this the Le Borgne-Rossin procedure. Thus, η^t contains stable sites that toppled at time t, and unstable sites that received grains. T^t contains for every site the total number of topplings up to time t. For every site $\mathbf{x}$, we then have

$$T^{t+1}(\mathbf{x}) = \max\{\left\lfloor \frac{1}{2d}\left(H^0(\mathbf{x}) + \sum_{\mathbf{y}\sim\mathbf{x}} T^t(\mathbf{y})\right)\right\rfloor, 0\}, \tag{5.4.14}$$

where $\mathbf{y}\sim\mathbf{x}$ means that $\mathbf{y}$ is a neighbor of $\mathbf{x}$. We remark that this formula should correspond to formula (2) of [7], but, apparently due to a misprint, there the term $H^0(\mathbf{x})$ was omitted.

If we divide the grid $\mathbb{Z}^d$ in two ("checkered") subgrids $\{\mathbf{x} : \sum_{i=1}^d x_i \text{ is even}\}$ and $\{\mathbf{x} : \sum_{i=1}^d x_i \text{ is odd}\}$, then at every time t, at most one subgrid contains unstable sites, because the neighbors of sites of one subgrid are all in the other subgrid, and we started with only one unstable site. We will consider next-nearest neighbor pairs $\mathbf{x}$ and $\mathbf{z}$, i.e., $\mathbf{x}$ and $\mathbf{z}$ are in the same subgrid, at most 2 coordinates differ, and $\sum_i ||x_i| - |z_i|| = 2$, or in other words, $\mathbf{z}$ is a neighbor of a neighbor of $\mathbf{x}$, $\mathbf{x}$ itself being excluded.

First, we will prove by induction in t that for every next-nearest neighbor pair $\mathbf{x}$ and $\mathbf{z}$, with $d(\mathbf{x}) \leq d(\mathbf{z})$, where $d(\mathbf{x})$ is the Euclidean distance of $\mathbf{x}$ to the origin, $T^t(\mathbf{x}) \geq T^t(\mathbf{z})$. We remark that for a next-nearest neighbor pair $\mathbf{x}'$ and $\mathbf{z}'$ with $d(\mathbf{x}') = d(\mathbf{z}')$, we must have that $\mathbf{x}'$ is equal to $\mathbf{z}'$ up to a permutation of coordinates. The model should be invariant under such a permutation, therefore, in that case we have $T^t(\mathbf{x}') = T^t(\mathbf{z}')$.

The statement is true at $t = 0$, since at $t = 0$, we have $T^0(\mathbf{x}) = 0$ for all $\mathbf{x}$. It is also true at $t = 1$, since at $t = 1$, only the origin topples.

Now suppose that the statement is true at time t, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two next-nearest neighbor sites that do not topple at time t. Suppose $d(\mathbf{x}_1) < d(\mathbf{x}_2)$. Then by assumption, $\sum_{\mathbf{y}_1 \sim \mathbf{x}_1} T^t(\mathbf{y}_1) \geq \sum_{\mathbf{y}_2 \sim \mathbf{x}_2} T^t(\mathbf{y}_2)$, because the neighbors of $\mathbf{x}_1$ and $\mathbf{x}_2$ can be grouped into next-nearest neighbor pairs with $d(\mathbf{y}_1) < d(\mathbf{y}_2)$. Furthermore, we always have $H^0(\mathbf{x}_1) \geq H^0(\mathbf{x}_2)$, since $H^0(\mathbf{0}) = n$, and $H^0(\mathbf{x}) = h$ if $\mathbf{x} \neq \mathbf{0}$.

Inserting this in (5.4.14), we obtain $T^{t+1}(\mathbf{x}_1) \geq T^{t+1}(\mathbf{x}_2)$, so that the statement remains true at $t + 1$ for one subgrid. The other subgrid must contain only sites that do not topple at $t + 1$, so that for all $\mathbf{x}$ in this subgrid, $T^{t+1}(\mathbf{x}) = T^t(\mathbf{x})$. Therefore, the statement remains true at $t + 1$ for both subgrids.

Now let $\mathbf{x}$ be a site with maximal $r = \max_i |x_i|$ where $T(\mathbf{x}) > 0$ (by symmetry, $\mathbf{x}$ cannot be unique). Then no sites outside $\mathfrak{C}(r)$ can have toppled. Furthermore, all next-nearest neighbors of $\mathbf{x}$ that are closer to the origin have also toppled, and subsequently all next-nearest neighbors of those next-nearest neighbors, that are still closer to the origin, etc. Then we use symmetry to find that all sites in $\mathfrak{D}(r)$ and in the same subgrid as $\mathbf{x}$ must also have toppled. An example for $d = 2$ is depicted in Figure 5.4.4.

·	·	z	·	z	·	·
·	y	·	z	·	z	·
$\mathbf{x}$	·	y	·	z	·	z
·	y	·	$\mathbf{0}$	·	z	·
z	·	z	·	z	·	z
·	z	·	z	·	z	·
·	·	z	·	z	·	·

Figure 5.4.4. Let x be a site with maximal $r = \max_i |x_i|$ that toppled. In this example, $r = 3$. By the fact that all next-nearest neighbors that are close to the origin have also toppled, we find that sites y, and the origin, have also toppled. By symmetry, we find that sites z have also toppled. Then we conclude that all sites in $\mathfrak{D}(3)$ (colored lightgrey) and in the same subgrid as x have toppled, and no sites outside $\mathfrak{C}(3)$ in the same subgrid can have toppled.

Furthermore, for at least one of the neighbors $\mathbf{y}$ of $\mathbf{x}$, we must have $T(\mathbf{y}) > 0$, because $\mathcal{T}_n$ is path connected. This neighbor has $\max_i |y_i| \geq r - 1$. For this neighbor, we can make the same observation. It follows that all sites in $\mathfrak{D}(r - 1)$ have toppled, so that the first part of the theorem follows. $\square$

To prove the inequalities for r_n, we need the following lemma:

Lemma 5.4.15. *Let ρ_n be the average value of $H_n(\mathbf{x})$ in $\mathcal{T}_n$, σ_n the number of sites and β_n the number of internal bonds in $\mathcal{T}_n$. Then*

$$\frac{\beta_n}{\sigma_n} \leq \rho_n \leq 2d - 1.$$

Proof. It suffices to prove that the configuration restricted to $\mathcal{T}_n$ is recurrent [10, 39]. This can be checked in a nonambiguous way with the burning algorithm [10]. The inequality then follows from the fact that in a recurrent, stable configuration restricted to some set, the total number of particles in the set is at least the number of internal bonds in the set, and at most h_{max} times the number of sites in the set. In the case of an unstable configuration, the term "recurrent" is somewhat inappropriate, but we still define it to indicate configurations that pass the burning algorithm.

We use the following properties of recurrent configurations:

1. recurrence of a configuration restricted to a set is conserved under addition of a particle to the set, and under a legal toppling in the set.

2. if the configuration restricted to a set is recurrent, and we add an unstable site $\mathbf{x}$ to the set, then the configuration is recurrent restricted to the extended set.

We will use induction in n. We will show that if η_n restricted to $\mathcal{T}_n$ is recurrent, then η_{n+1} restricted to $\mathcal{T}_{n+1}$ is recurrent. For a starting point of the induction, we choose $n' = 2d - h$, so that $|\mathcal{T}_{n'}| = 1$, because a configuration restricted to a single site is always recurrent.

We can obtain η_{n+1} by starting from η_n, adding a particle to the origin, and stabilizing. If we first stabilize restricted to $\mathcal{T}_n$, then by the first property of recurrence, the configuration restricted to $\mathcal{T}_n$ is still recurrent. If $\mathcal{T}_n = \mathcal{T}_{n+1}$, then we are now done. But it is also possible that at this point, there are unstable sites outside $\mathcal{T}_n$.

By both properties of recurrence, if we add one of these sites to $\mathcal{T}_n$ and stabilize with respect to the new set, the configuration restricted to the new set is still recurrent. We can repeat this step until there are no more unstable sites to be added. Then the new set is $\mathcal{T}_{n+1}$, and the new configuration is η_{n+1}. □

Proof of Theorem 5.4.11, part (2). To calculate the inequalities for r_n, we observe that the minimum r_n would be obtained if $\mathcal{V}_n$ would equal the cube $\mathfrak{C}(r_n + 1)$, with each site containing the maximal number of particles. The number of sites in this cube is $(2r_n + 3)^d$, and the number of particles per site is then $2d - 1 - h$, so that

$$n \leq (2r_n + 3)^d (2d - 1 - h),$$

for every n.

The maximal r_n would be obtained if $\mathcal{V}_n$ would equal the diamond $\mathfrak{D}(r_n)$, containing the minimum number of particles. As we deduced that the configuration restricted to $\mathcal{T}_n$ is recurrent, we calculate the minimum average value of $H(\mathbf{x})$ on $\mathfrak{D}(r_n - 1)$, using Lemma 5.4.15. The number of internal bonds in $\mathfrak{D}(r_n - 1)$ is $(2r_n - 2)^d$. The number of sites in $\mathfrak{D}(r_n - 1)$ is $\frac{1}{d}(2r_n - 1)^d + o(r_n^d)$ as $r_n \to \infty$. Thus,

$$n \geq (2r_n - 2)^d - \frac{h}{d}(2r_n - 1)^d + o(r_n^d), \qquad r_n \to \infty.$$

As in the limit $n \to \infty$ we also have $r_n \to \infty$, this leads to the second inequality for r_n in the theorem. □

The proof of Theorem 5.4.11 allows to prove an interesting characteristic of $\mathcal{V}_n$:

Proposition 5.4.16. *For the sandpile model, $\mathcal{V}_n$ is simply connected for all n.*

Proof. We first prove that $\mathcal{T}_n$ is simply connected, by contradiction.

Suppose that $\mathcal{T}_{n,SP,h}$ is not simply connected, that is, $\mathcal{T}_{n,SP,h}$ contains holes. This means there must be a site $\mathbf{y}$ in $\mathcal{T}_{n,SP,h}$ "beyond a hole", that is, a site y with $T(\mathbf{y}) > 0$, and at least one neighbor $\mathbf{x}$ of $\mathbf{y}$ with $T(\mathbf{x}) = 0$, that is closer to the origin than y itself and all neighbors x' of y for which $T(x') > 0$. At least one neighbor $\mathbf{x}'$ of $\mathbf{y}$ must have $T(\mathbf{x}') > 0$, because $\mathcal{T}_n$ is path connected. Thus, among the neighbors of $\mathbf{y}$, there must be a next-nearest neighbor pair $\mathbf{z}$ and $\mathbf{z}'$, with $d(\mathbf{z}') \geq d(\mathbf{z})$, $T(\mathbf{z}) = 0$ and $T(\mathbf{z}') > 0$. But this contradicts the above derived property that for all t, for every next-nearest neighbor $\mathbf{z}$ and $\mathbf{z}'$ with $d(\mathbf{z}) \geq d(\mathbf{z}')$, $T^t(\mathbf{z}) \leq T^t(\mathbf{z}')$, as this property should also hold for the final configuration.

The above does not suffice to conclude that $\mathcal{V}_n$ is also simply connected. Since $\mathcal{V}_n = \mathcal{T}_n \cup \delta\mathcal{T}_n$, we must show that T_n does not contain so-called *fjords*, i.e., places where the boundary of $\mathcal{T}_n$ nearly touches itself, such that $\mathcal{T}_n \cup \delta\mathcal{T}_n$ would contain a hole. But when one supposes that $\mathcal{T}_n$ does contain a fjord, then one can derive the same contradiction as above. Therefore, $\mathcal{V}_n$ is simply connected. □

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Chapter 6

Samenvatting

Zandhoopmodellen: het oneindig volume model, Zhang's model en limietvormen.

Dit proefschrift bevat vier artikelen die tot stand gekomen zijn tijdens dit promotieonderzoek, over diverse aspecten van zandhoopmodellen.

Zandhoopmodellen zijn dynamische modellen, die evolueren in discrete tijd. Ze zijn gedefinieerd op een eindig rooster; in dit proefschrift bekijken we alleen het rooster $\mathbb{Z}^d$ en eindige subsets daarvan. Een zandhoopmodel start vanuit een beginconfiguratie die bestaat uit een hoogte, of aantal zandkorrels, op ieder roosterpunt. De beginconfiguratie moet stabiel zijn, dat wil zeggen dat op ieder punt de hoogte onder een bepaalde grenswaarde moet zijn. Een tijdstap bestaat uit een toevoeging en daaropvolgend de stabilisatie. De toevoeging is aan een willekeurig roosterpunt; de hoogte neemt daar toe, meestal met een zandkorrel, maar in Zhang's model bijvoorbeeld met een random hoeveelheid zand. Stabilisatie gebeurt door middel van topplings van instabiele punten: ieder roosterpunt waarvan de hoogte op of boven de grenswaarde is, is instabiel en moet zijn hoogte verminderen door zand aan de naburige punten te geven. Bij een toppling van een punt aan de rand van het rooster verdwijnt er ook zand naar de "ontbrekende buren", d.w.z., over de rand. Toppelen gaat door tot de configuratie weer stabiel is. Dan is de nieuwe configuratie bereikt, en kan de volgende tijdstap beginnen.

Bak, Tang and Wiesenfeld introduceerden in [5] een zandhoopmodel als model voor "self-organized criticality". Deze term bedachten ze voor het voorkomen van kenmerken van criticaliteit, zoals zware staart- verdelingen en lange afstand-correlaties, in diverse natuurlijke fenomenen als aardbevingen of bosbranden. In statistische mechanica, waar modellen gedefinieerd zijn in oneindig volume, is criticaliteit goed bekend; de kritische toestand wordt bereikt bij de kritieke waarde van een modelparameter, waar een fase-overgang optreedt. Een voorbeeld van zo'n parameter is de temperatuur in het Ising model. Het idee bij self-organized criticality is dat het model spontaan de kritische toestand bereikt, zonder dat er sprake is van een modelparameter.

De hoofdstukken 2 en 3 van dit proefschrift gaan over het abelse zandhoopmodel. In dit model bestaat een toevoeging uit 1 zandkorrel. Punten met een hoogte van minimaal $2d$ zijn instabiel. Als een punt toppelt verliest het $2d$ korrels en ieder naburig punt krijgt er een bij. Dit zandhoopmodel wordt abels genoemd vanwege de eigenschap dat de bereikte configuratie onafhankelijk is van de volgorde van toppelen.

Het doel in deze hoofdstukken is het vinden van een relatie tussen self-organized criticality in het abelse zandhoopmodel, en criticaliteit zoals bekend uit statistische mechanica. Daartoe hebben we een oneindig volume zandhoopmodel gedefinieerd, waarin een parameter voorkomt. Het idee van deze aanpak is dat we een kritieke waarde van de parameter zoeken, en dan de dynamica van het oorspronkelijke zandhoopmodel interpreteren als een mechanisme wat deze parameter naar de kritieke waarde stuurt.

Dit oneindig volume model start vanuit een -niet noodzakelijk stabiele- beginconfiguratie, gekozen volgens een translatie-invariante beginmaat op het oneindig rooster $\mathbb{Z}^d$. Deze configuratie evolueert in de tijd door het toppelen van instabiele punten. Het kiezen van een toppelvolgorde is nu minder triviaal; in hoofdstuk 3 introduceren we toppelprocedures, dat zijn "toegestane" volgordes. Voor sommige beginconfiguraties is er dan een stabiele eindconfiguratie na maar eindig veel topplings per punt; die noemen we stabiliseerbaar. Voor andere beginconfiguraties blijft het aantal topplings per punt toenemen, welke (toegestane) toppelvolgorde we ook kiezen. We noemen een beginmaat stabiliseerbaar als (vrijwel) iedere configuratie volgens deze maat dat is. Een parameter in dit model is het volgens de beginmaat verwachte aantal korrels per punt, ofwel de dichtheid. We vinden dat beginmaten met dichtheid kleiner dan d altijd, en met dichtheid groter dan $2d-1$ nooit stabiliseerbaar zijn. In hoofdstuk 2 is dit bewezen voor twee specifieke toppelprocedures, in hoofdstuk 3 voor alle. Voor elke dichtheid tussen deze waarden bestaan stabiliseerbare en niet-stabiliseerbare beginmaten. Voor $d = 1$ vallen deze twee waarden samen, in dat geval is er een stabiliseerbaarheids-faseovergang. We bewijzen dat productmaten met de kritieke dichtheid 1 niet stabiliseerbaar zijn. We kijken voor dit model, voor stabiliseerbare beginmaten, ook naar percolatie van punten die getoppeld hebben tijdens stabilisatie. Het blijkt dat als de dichtheid klein genoeg is -maar niet 0-, de clusters van getoppelde punten eindige grootte hebben.

Hoofdstuk 4 gaat over Zhang's model. In dit zandhoopmodel bestaat een toevoeging uit een random hoeveelheid zand, uniform verdeeld op $[a, b] \subseteq [0, 1]$. De grenswaarde voor de hoogte is 1. Als een punt toppelt verliest het zijn gehele hoogte; een fractie $\frac{1}{2d}$ gaat naar elk van de naburige punten. Hoewel voor dit model in principe op elk punt iedere hoogte tussen 0 en 1 voor kan komen, blijkt uit simulaties voor grote roosters, dat de stationaire verdeling voor de hoogte per punt geconcentreerd is rond $2d$ equidistante waarden (alleen in het geval $d = 1$ is er maar een zo'n waarde). Zhang noemde die waarden "quasi-units". Zhang's model is niet abels. Hoofdstuk 4 begint met het introduceren van het model in dimensie 1 op N punten, en een uitgebreide bespreking van overeenkomsten en verschillen met het abelse zandhoopmodel. Het voornaamste resultaat is voor $a \geq \frac{1}{2}$, in de limiet van $N \to \infty$: we bewijzen dat de stationaire verdeling voor de hoogte per punt concentreert rond de waarde $\frac{a+b}{2}$, ofwel, een bevestiging van het bestaan van Zhang's quasi-units voor dit geval. Verder vinden we voor het geval $[a, b] = [0, 1]$, onder enkele aannames, dat de verwachte stationaire hoogte gelijk is aan $\sqrt{1/2}$, en we geven een expliciete uitdrukking voor de stationaire verdeling voor het model op één punt.

Het laatste hoofstuk, hoofdstuk 5, gaat over het abelse zandhoopmodel als deterministisch groeimodel. De beginconfiguratie bestaat uit h zandkorrels op alle punten van $\mathbb{Z}^d$. Als h negatief is, kun je dit voorstellen als een kuiltje op ieder punt waar $|h|$ korrels in passen. Er wordt alleen aan de oorsprong toegevoegd; we kiezen $h \leq 2d - 2$ om ervoor te zorgen dat de topplings binnen een eindig gebied blijven. We zijn geinteresseerd in de vorm het gebied waarover het toegevoegde zand zich verspreidt, en of er een limietvorm bestaat. Voor het gerelateerde rotor-router model [] met $h = -1$ is de limietvorm een bol. Voornaamste resultaten van hoofdstuk 5 zijn: de limietvorm voor het zandhoopmodel met $h = 2d - 2$ is een kubus, de limietvorm voor het rotor-router model is een bol voor elke $h \leq -1$, en de limietvorm voor het zandhoopmodel is een bol als $h \to \infty$.

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